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NAHM TRANSFORM FOR INTEGRABLE CONNECTIONS ON THE RIEMANN SPHERE

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Abstract. – In this text, we define Nahm transform for parabolic integrable connections with regular singularities and one Poincaré rank 1 irregular singularity on the Riemann sphere. After a first definition using L^2-cohomology, we give an algebraic description in terms of hypercohomology. Exploiting these different interpretations, we give the transformed object by explicit analytic formulas as well as geometrically, by its spectral curve. Finally, we show that this transform is (up to a sign) an involution.

Résumé **(Transformée de Nahm pour les connexions intégrables sur la sphère de Riemann)**
Dans ce texte, nous définissons la transformée de Nahm pour les connexions intégrables paraboliques ayant des singularités régulières et une singularité irrégulière de rang de Poincaré 1 sur la sphère de Riemann. Après une définition en terme de cohomologie L^2, nous donnons une description algébrique en terme d'hypercohomologie. En nous servant de cette double interprétation, nous décrivons l'objet transformé à la fois par des formules analytiques explicites et géométriquement en utilisant la courbe spectrale du problème. Finalement, nous démontrons que la correspondance définie est (à un signe près) une involution.

CONTENTS

INTRODUCTION

Nahm transform is a non-linear analog for instantons of the usual Fourier transform on functions. It has been extensively studied starting from the beginning of the 1980's, inspired by the seminal work of M. F. Atiyah, V. Drinfeld, N. J. Hitchin and Yu. I. Manin on a correspondence (the *ADHM-transform*) between finite-energy solutions of the *Yang-Mills equations* and some algebraic data (see **[1]** and Chapter 3 of **[12]**). The Yang-Mills equations are the anti-self-duality equations for a unitary connection on a Hermitian vector bundle defined over $\mathbf{R}^4$; their finite-energy solutions are called *instantons*.

Since then, it turned out that the general picture concerning this correspondence is as follows: let X be any manifold obtained as a quotient of $\mathbf{R}^4$ by a closed additive subgroup Λ. The solutions of the Yang-Mills equations invariant by Λ (that are clearly not of finite energy in the case $\Lambda \neq \{0\}$) can be identified in an obvious manner to solutions of a system of differential equations on X, called the *reduction* of the Yang-Mills equations. On the other hand, denoting by $(\mathbf{R}^4)^*$ the dual of the vector space $\mathbf{R}^4$, Λ determines a closed additive subgroup Λ^* called the *dual subgroup* by saying that an element $\xi \in (\mathbf{R}^4)^*$ is in Λ^* if and only if $\xi(\lambda) \in \mathbf{Z}$ for all $\lambda \in \Lambda$. Hence, we can form the *dual manifold* $X^* = (\mathbf{R}^4)^*/\Lambda^*$ of X, that also admits a reduction of the Yang-Mills equations. Nahm transform is then a procedure that maps solutions of the reduced equations on X to solutions of the reduced equations on X^* bijectively up to overall gauge transformations on both sides. One remarks that there is a canonical isomorphism between $((\mathbf{R}^4)^*)^*$ and $\mathbf{R}^4$, as well as between $(\Lambda^*)^*$ and Λ. Therefore, if we start from a solution of the reduced equations on X and iterate Nahm transform twice, we again get a solution of the reduced equations on X. One important property analogous to usual Fourier transform is that in some cases the solution we get this way is, up to a coordinate change $x \mapsto -x$, known to be the solution we started with; that is, Nahm transform is (up to a sign) *involutive*. Moreover, in some cases one knows that the moduli spaces of solutions of the reduced equations modulo gauge transformations on X and on X^* are smooth hyper-Kähler manifolds with respect to the metric induced by L^2-norm and the complex structures induced by $\mathbf{R}^4$; Nahm transform is then a hyper-Kähler isometry between these moduli spaces. This is to be compared with Parseval's theorem which states that usual Fourier transform defines an isometry between L^2-spaces of functions.

Putting $\Lambda = \{0\}$, one gets $X = \mathbf{R}^4$ and $\Lambda^* = \mathbf{R}^4$, so $X^* = \{0\}$. In this case, Nahm transform reduces to the ADHM-transform. The other examples of Nahm transform in the literature for different subgroups of $\mathbf{R}^4$ are as follows. For $\Lambda = \mathbf{Z}^4$, staring from an ASD-connection on the four-dimensional torus $X = T^4$, its transform is an ASD-connection on the dual torus $X^* = (T^4)^*$, see P. Braam and P. van Baal [**7**], S. Donaldson and P. Kronheimer [**12**] and H. Schenck [**25**]. Notice that [**12**] also describes a holomorphic interpretation of this transform, which reproduces Mukai's Fourier transform for holomorphic bundles on tori. For $X = \mathbf{R}^3, X^* = \mathbf{R}$ one gets a correspondence between monopoles (solutions of Bogomolny's equation on $\mathbf{R}^3$) and solutions of an ordinary differential equation, called Nahm's equation, on the open interval $(-1, 1)$, with fixed singularity behaviour at the end-points. This was first described by W. Nahm [**21**], then complemented by others. The case $X = \mathbf{R}^2 \times S^1, X^* = \mathbf{R} \times S^1$ was treated by S. Cherkis and A. Kapustin [**10**]: here, one gets a correspondence between periodic monopoles on $\mathbf{R}^2 \times S^1$ with logarithmic growth at infinity and solutions of Hitchin's equations on $\mathbf{R} \times S^1$ with exponential growth at infinity. When $X = \mathbf{R}^3 \times S^1, X^* = S^1$, the correspondence relates calorons (periodic instantons) on $\mathbf{R}^3 \times S^1$ and solutions of Nahm's equations on the circle with singularity in a discrete set of points. This was studied by T. Nye [**22**] and T. Nye and M. Singer [**23**]. In these works invertibility is not yet completely proved; however, J. Hurtubise and B. Charbonneau recently announced [**9**] that they completed its proof. In the case $X = \mathbf{R}^2 \times T^2$ the works of M. Jardim [**16**], [**17**] and O. Biquard and M. Jardim [**6**] establish the transform between doubly-periodic instantons (ASD-connections on $\mathbf{R}^2 \times T^2$) with fixed behaviour at infinity, and solutions of Hitchin's equations on $X^* = T^2$ with (at most) two simple poles and fixed singularity data. Finally, for $X = \mathbf{R} \times T^3$, B. Charbonneau described a transform from spatially periodic instantons to singular monopoles on $X^* = T^3$ [**8**]. For more details on the history of these examples, see the survey paper [**18**] of M. Jardim.

In this work, we are concerned with one of the last cases not treated before, namely $\Lambda = \mathbf{R}^2$. In this case, the base manifold is $X = \mathbf{R}^2$, and its dual X^* is another copy of the real plane that we shall denote by $\hat{\mathbf{R}}^2$. These are non-compact manifolds, with compactifications the Riemann spheres $\mathbf{CP}^1$ and $\widehat{\mathbf{CP}}^1$ respectively. The reduction of the original (Yang-Mills) equations can be viewed in two different ways depending on the complex structure that we choose: they are the equations defining an integrable connection with harmonic metric, or equivalently, those defining a Higgs bundle with Hermitian-Einstein metric. Now, it turns out that there are no smooth solutions on the Riemann sphere of either one of these equations except for the trivial ones (cf. [**14**]). However, there are solutions with prescribed singularities in some points, and the solutions of one equation are still in correspondence with those of the other: this is proved by O. Biquard and Ph. Boalch in [**5**]. For this correspondence to work, one needs to have a parabolic structure in the singular locus on both types of objects. We establish, under some hypotheses on the singularity behaviour, Nahm transform for parabolic integrable connections (or equivalently, parabolic Higgs bundles) on the