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STABILITY AND INSTABILITY IN NINETEENTH-CENTURY FLUID MECHANICS

by Olivier Darrigol (*)

> There is scarcely any question in dynamics more important for Natural Philosophy than the stability of motion.
>
> W. Thomson and P.G.Tait [1867, § 346]

ABSTRACT. — The stability or instability of a few basic flows was conjectured, debated, and sometimes proved in the nineteenth century. Motivations varied from turbulence observed in real flows to permanence expected in hydrodynamic theories of matter. Contemporary mathematics often failed to provide rigorous answers, and personal intuitions sometimes gave wrong results. Yet some of the basic ideas and methods of the modern theory of hydrodynamic instability occurred to the elite of British and German mathematical physics, including Stokes, Kelvin, Helmholtz, and Rayleigh. This usually happened by reflecting on concrete specific problems, with a striking variety of investigative styles.

RÉSUMÉ. — STABILITÉ ET INSTABILITÉ EN MÉCANIQUE DES FLUIDES AU XIX^{e} SIÈCLE. — Au dix-neuvième siècle, la stabilité ou l'instabilité de quelques écoulements simples fut l'objet de conjectures, de débats et parfois de preuves mathématiques. Les motivations pour ce type de recherche variaient considérablement, de la turbulence observée d'écoulements réels à la permanence attendue dans les théories hydrodynamiques de la matière. Les mathématiques contemporaines étaient rarement en mesure de fournir des réponses rigoureuses et les intuitions des uns et des autres conduisirent parfois à des résultats faux. Néanmoins, quelques grands de la physique mathématique britannique et allemande — Stokes, Kelvin, Helmholtz et Rayleigh — développèrent certaines idées et méthodes de base de la théorie moderne des instabilités hydrodynamiques. Ils y parvinrent en réfléchissant à des problèmes spécifiques concrets, avec une étonnante diversité de styles.

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Olivier DARRIGOL, CNRS: Rehseis, Université Paris VII, Centre Javelot, 2 place Jussieu, 75251 Paris Cedex 05. Courrier électronique: darrigol@paris7.jussieu.fr
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Instability of motion haunted celestial mechanics from the beginning of Newtonian theories. In the nineteenth century, it became a central question of the developing fluid mechanics, for two reasons. Firstly, the discrepancy between actual fluid behavior and known solutions of the hydrodynamic equations suggested the instability of these solutions. Secondly, the British endeavor to reduce all physics to the motion of a perfect liquid presupposed the stability of the forms of motion used to describe matter and ether. Instability in the former case, stability in the latter needed to be proved.

In nineteenth-century parlance, kinetic instability broadly meant departure from an expected regularity of motion. In hydrodynamics alone, it included unsteadiness, non-uniqueness of motion, sensibility to infinitesimal local perturbation, sensibility to infinitesimal harmonic perturbations, sensibility to finite perturbations, sensibility to infinitely small viscosity. This spectrum of meanings is much wider than a modern treatise on hydrodynamic stability would tolerate. A narrower selection would not befit a historical study, for it would artificially separate issues that nineteenth-century writers conceived as a whole.

The first section of this paper is devoted to George Stokes' pioneering emphasis on hydrodynamic instability as the probable cause of the failure of Eulerian flows to reproduce essential characteristics of the observed motions of slightly viscous fluids (air and water). Stokes believed instability to occur whenever the lines of flow diverged too strongly, as happens in a suddenly enlarged conduit or past a solid obtacle. The second section recounts how Hermann Helmholtz (1868) and William Thomson (1871) introduced another type of instability, now called the Kelvin-Helmholtz instability, following which the discontinuity surface between two adjacent parallel flows of different velocities loses its flatness under infinitesimal perturbation. Helmholtz thus explained the instability of a jet of a fluid through a stagnant mass of the same fluid, for instance the convolutions of the smoke from a chimney. Thomson's motivation was the theory of wave formation on a water surface under wind.

In the Helmholtz-Kelvin case, instability was derived from the hydrodynamic equations. In Stokes' case, it only was a conjecture. Yet the purpose was the same: to save the phenomena. In contrast, Thomson's vortex theory of matter required stability for the motions he imagined in the primitive perfect liquid of the world. This theory, which began in 1867, is discussed in the third section of this paper. Thomson could only prove the stability of motions simpler than those he needed. For many years, he contented himself with an analogy with the observed stability of smoke rings. At last, in the late 1880s, he became convinced that vortex rings were unstable.

Owing to their different interests, Stokes and Thomson had opposite biases about hydrodynamic (in)stability. This is illustrated in the fourth section of this paper, through an account of their long, witty exchange on the possibility of discontinuity surfaces (infinitely thin layers of infinite shear) in a perfect liquid. From his first paper (1842) to his last letter to Thomson (1901), Stokes argued that the formation of surfaces of discontinuity provided a basic mechanism of instability for the flow of a perfect liquid past a solid obstacle. Thomson repeatedly countered that such a process would violate fundamental hydrodynamic theorems and that viscosity played an essential role in Stokes' instabilities. The two protagonists never came to an agreement, even though they shared many cultural values within and without physics.

The fifth section of this paper deals with the (in)stability of parallel flow. The most definite nineteenth-century result on this topic was Lord Rayleigh's criterion of 1880 for the stability of two-dimensional parallel motion in a perfect liquid. The context was John Tyndall's amusing experiments on the sound-triggered instability of smoke jets. However, the strongest motivation for theoretical inquiries in parallel-flow stability was Osborne Reynolds' precise experimental account (1883) of the transition between laminar and turbulent flow in the case of circular pipes. In 1887 Cambridge authorities, including Stokes and Rayleigh, made the theory of this transition the topic of the Adams prize for 1889. This prompted Thomson to publish proofs of instability for two cases of parallel, two-dimensional viscous flow. Rayleigh soon challenged these proofs. William Orr proved their incompleteness in 1907.

In sum, the nineteenth-century concern with hydrodynamic stability led to well-defined, clearly stated questions on the stability of the solutions of the fundamental hydrodynamic equations (Euler and Navier-Stokes). Most answers to these questions were tentative, controversial, or plainly wrong. The subject that Rayleigh judged "second to none in scientific as well as practical interest" [*RSP* 2, p. 344] remained utterly confused. Besides the Helmholtz-Kelvin instability and Rayleigh's inflection theorem, the theoretical yield was rather modest: there was Stokes' vague, unproved instability of divergent flows, Thomson's unproved instability of vortex rings, the hanging question of the formation of discontinuity surfaces, and two illusory proofs of stability for simple cases of parallel viscous flow.

The situation could be compared to number theory, which is reputed for the contrast between the simple statements of some of its problems and the enormous difficulty of their solution. The parallel becomes even stronger if we note that some nineteenth-century questions on hydrodynamic stability, for example the stability of viscous flow in circular pipes or the stability of viscous flow past obstacles are yet to be answered, and that the few available answers to such questions were obtained at the price of considerable mathematical efforts. This long persistence of basic questions of fluid mechanics is the more striking because in physics questions tend to change faster than their answers.

In number theory, failed demonstrations of famous conjectures sometimes brought forth novel styles of reasoning, interesting side-problems, and even new branches of mathematics. Something similar happened in the history of hydrodynamic stability, though to a less spectacular extent. Stokes' and Helmholtz's surfaces of discontinuity were used to solve the old problem of the *vena contracta* and to determine the shape of liquid jets [Kirchhoff 1869], [Rayleigh 1876]. They also permitted Rayleigh's solution (1876) of d'Alembert's paradox, and inspired some aspects of Ludwig Prandtl's boundary-layer theory (1904). Rayleigh's formulation of the stability problem in terms of the real or imaginary character of the frequency of characteristic perturbation modes is the origin of the modern method of normal modes [Drazin and Reid 1981, pp. 10–11].

As a last important example of fruitful groping, Stokes, Thomson, and Rayleigh all emphasized that the zero-viscosity limit of viscous-fluid behavior could be singular. Stokes regarded this singularity as a symptom of instability of inviscid, divergent flows; Thomson as an indication that the formation of unstable states of parallel motion required finite viscosity; Rayleigh as a clue to why some states of parallel motion were stable for zero viscosity and unstable for small, finite viscosity. Rayleigh [1892,

p. 577] even anticipated the modern concept of boundary-layer instability:

"But the impression upon my mind is that the motions calculated above for an absolutely inviscid liquid may be found inapplicable to a viscid liquid of vanishing viscosity, and that a more complete treatment might even yet indicate instability, perhaps of a local character, in the immediate neighbourhood of the walls, when the viscosity is very small."

In the absence of mathematical proof, the value of such utterances may be questioned. Rayleigh himself [1892, p. 576] warned that "speculations on such a subject in advance of definite arguments are not worth much." Many years later, Garrett Birkhoff [1950] reflected that speculations were especially fragile on systems like fluids that have infinitely many degrees of freedom. Yet by imagining odd, singular behaviors, the pioneers of hydrodynamics instability avoided the temptation to discard the foundation of the field, the Navier-Stokes equation; and they sometimes indicated fertile directions of research.

In sum, early struggles with hydrodynamic stability are apt to inform the later history of this topic. They also reveal fine stylistic differences among some of the leading physicists of the nineteenth century. In the lack of rigorous mathematical reasoning, these physicists had to rely on subtle, personal combinations of intuition, past experience or experiment, and improvised mathematics. They ascribed different roles to idealizations such as inviscidity, rigid walls, or infinitely sharp edges. For instance, Helmholtz and Stokes believed that the perfect liquid provided a correct intuition of low-viscosity liquid behavior, if only discontinuity surfaces were admitted. Thomson denied that, and reserved the perfect liquid (without discontinuity) for his sub-dynamics of the universe. As the means lacked to exclude rigorously one of these two views, the protagonists preserved their colorful identities.

In the following, vector notation is used anachronistically for the sake of concision. Following Thomson's convention, by perfect liquid is meant an incompressible, inviscid fluid. In order that the present paper may be read independently, some sections of earlier papers of mine (on Helmholtz's surfaces of discontinuity and on Reynolds' study of pipe flow) are reproduced in abbreviated form.