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A GUIDE TO THE REDUCTION MODULO p OF SHIMURA VARIETIES

by

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Abstract. — This is a survey of recent work on the reduction of Shimura varieties with parahoric level structures.

***Résumé* (Un guide à la réduction modulo p des variétés de Shimura).** — Cet article est un survol de resultats sur la réduction des variétés de Shimura à structure de niveau parahorique.

This report is based on my lecture at the Langlands conference in Princeton in 1996 and the series of lectures I gave at the semestre Hecke in Paris in 2000. In putting the notes for these lectures in order, it was my original intention to give a survey of the activities in the study of the reduction of Shimura varieties. However, I realized very soon that this task was far beyond my capabilities. There are impressive results on the reduction of "classical" Shimura varieties, like the Siegel spaces or the Hilbert-Blumenthal spaces, there are deep results on the reduction of specific Shimura varieties and their application to automorphic representations and modular forms, and to even enumerate all these achievements of the last few years in one report would be very difficult. Instead, I decided to concentrate on the reduction modulo p of Shimura varieties *for a parahoric level structure* and more specifially on those aspects which have a *group-theoretic interpretation*. Even in this narrowed down focus it was not my aim to survey all results in this area but rather to serve as a guide to those problems with which I am familiar, by putting some of the existing literature in its context and by pointing out unsolved questions. These questions or conjectures are of two different kinds. The first kind are open even for those Shimura varieties which are moduli spaces of abelian varieties. Surely these conjectures are the most urgent

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and the most concrete and the most tractable. The second kind are known for these special Shimura varieties. Here the purpose of the conjectures resp. questions is to extend these results to more general cases, *e.g.* to Shimura varieties of Hodge type.

As a general rule, I wish to stress that I would not be surprised if some of the conjectures stated here turn out to be false, especially in cases of very bad ramification. But I believe that even in these cases I should not be too far off the mark, and that a suitable modification of these conjectures gives the correct answer. My motivation in running the risk of stating precise conjectures is that I wanted to point out directions of investigation which seem promising to me.

The guiding principle of the whole theory presented here is to give a group-theoretical interpretation of phenomena found in special cases in a formulation which makes sense for a general Shimura variety. This is illustrated in the first section which treats some aspects of the elliptic modular case from the point of view taken in this paper. The rest of the article consists of two parts, the local theory and the global theory. Their approximate contents may be inferred from the table of contents below.

I should point out that the development in these notes is very uneven and that sometimes I have gone into the nitty gritty detail, whereas at other times I only give a reference for further developments. My motivation for this is that I wanted to give a real taste of the whole subject — in the hope that it is attractive enough for a student, one motivated enough to read on and skip parts which he finds unappealing.

In conclusion, I would like to stress, as in the introduction of [**R2**], the influence of the ideas of V. Drinfeld, R. Kottwitz, R. Langlands and T. Zink on my way of thinking about the circle of problems discussed here. In more recent times I also learned enormously from G. Faltings, A. Genestier, U. Görtz, T. Haines, J. de Jong, E. Landvogt, G. Laumon, B.C. Ngô, G. Pappas, H. Reimann, H. Stamm, and T. Wedhorn, but the influence of R. Kottwitz continued to be all-important. I am happy to express my gratitude to all of them. I also thank T. Ito, R. Kottwitz and especially T. Haines for their remarks on a preliminary version of this paper. I am grateful to the referee for his careful reading of the paper and his helpful remarks.

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Contents

1. Motivation: The elliptic modular curve

In this section we illustrate the problem of describing the reduction modulo p of a Shimura variety in the simplest case. Let $\boldsymbol{G} = GL_2$ and let $(\boldsymbol{G}, \{h\})$ be the usual Shimura datum. Let $\mathbf{K} \subset \boldsymbol{G}(\mathbf{A}_f)$ be an open compact subgroup of the form $\mathbf{K} = K^p \cdot K_p$ where K^p is a sufficiently small open compact subgroup of $\boldsymbol{G}(\mathbf{A}_f^p)$. Let $G = \boldsymbol{G} \otimes_{\mathbf{Q}} \mathbf{Q}_p$. We consider the cases where K_p is one of the following two parahoric subgroups of $G(\mathbf{Q}_p)$,

(i) $K_p = K_p^{(i)} = GL_2(\mathbf{Z}_p)$ (*hyperspecial maximal parahoric*)
(ii) $K_p = K_p^{(ii)} = \{g \in GL_2(\mathbf{Z}_p);\ g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \bmod p\}$ (*Iwahori*)

The corresponding Shimura variety $\mathrm{Sh}(\boldsymbol{G}, h)_{\mathbf{K}}$ is defined over $\mathbf{Q}$. It admits a model $\mathrm{Sh}(G, h)_{\mathbf{K}}$ over $\mathrm{Spec}\,\mathbf{Z}_{(p)}$ by posing the following moduli problem over $(Sch/\mathbf{Z}_{(p)})$:

(i) an elliptic curve E with a level-K^p-structure.
(ii) an isogeny of degree p of elliptic curves $E_1 \to E_2$, with a level-K^p-structure.

The description of the point set $\mathrm{Sh}(G, h)_{\mathbf{K}}(\overline{\mathbf{F}}_p)$ takes in both cases (i) and (ii) the following form,

$$\mathrm{Sh}(G, h)_{\mathbf{K}}(\overline{\mathbf{F}}_p) = \coprod_{\varphi} I_\varphi(\mathbf{Q}) \backslash X(\varphi)_{K_p} \times X^p/K^p. \tag{1.1}$$

Here the sum ranges over the isogeny classes of elliptic curves and $I_\varphi(\mathbf{Q}) = \mathrm{End}_{\mathbf{Q}}(E)^\times$ is the group of self-isogenies of any element of this isogeny class. Furthermore, X^p/K^p may be identified with $\boldsymbol{G}(\mathbf{A}_f^p)/K^p$, with the action of $I_\varphi(\mathbf{Q})$ defined by the ℓ-adic representation afforded by the rational Tate module. The set $X(\varphi)_{K_p}$ is the most interesting ingredient.

Let $\mathcal{O} = W(\overline{\mathbf{F}}_p)$ be the ring of Witt vectors over $\overline{\mathbf{F}}_p$ and $L = \mathrm{Fract}\,\mathcal{O}$ be its fraction field. We denote by σ the Frobenius automorphism of L. Let N denote the rational Dieudonné module of E. Then N is a 2-dimensional L-vector space, equipped with a σ-linear bijective endomorphism F (the crystalline Frobenius). Then in case (i) (hyperspecial case), the set $X(\varphi)_{K_p^{(i)}}$ has the following description

$$
X(\varphi)_{K_p^{(i)}} = \{\Lambda;\ p\Lambda \subsetneqq F\Lambda \subsetneqq \Lambda\} \\
= \{\Lambda; \mathrm{inv}(\Lambda, F\Lambda) = \mu\}. \tag{1.2}
$$

Here Λ denotes a $\mathcal{O}$-lattice in N. The set of $\mathcal{O}$-lattices in N may be identified with $G(L)/G(\mathcal{O})$. We have used the elementary divisor theorem to establish an identification

$$
\mathrm{inv} : G(L)\backslash[G(L)/G(\mathcal{O}) \times G(L)/G(\mathcal{O})] = G(\mathcal{O})\backslash G(L)/G(\mathcal{O}) \simeq \mathbf{Z}^2/S_2. \tag{1.3}
$$

Furthermore $\mu = (1,0) \in \mathbf{Z}^2/S_2$ is the conjugacy class of one-parameter subgroups associated to $\{h\}$.

In case (ii) (Iwahori case), the set $X(\varphi)_{K_p^{(ii)}}$ has the following description,

$$
X(\varphi)_{K_p^{(ii)}} = \{p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2;\ p\Lambda_1 \subsetneqq F\Lambda_1 \subsetneqq \Lambda_1, p\Lambda_2 \subsetneqq F\Lambda_2 \subsetneqq \Lambda_2\}. \tag{1.4}
$$

Here again Λ_1, Λ_2 denote $\mathcal{O}$-lattices in N.

In either case $X(\varphi)_{K_p}$ is equipped with an operator Φ which under the bijection (1.1) corresponds to the action of the Frobenius automorphism on the left hand side.

Let us describe the set $X(\varphi)_{K_p^{(ii)}}$ in the manner of the second line of (1.2). The analogue in this case of the relative position of two chains of inclusions of $\mathcal{O}$-lattices in N, $p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2$ and $p\Lambda_2' \subsetneqq \Lambda_1' \subsetneqq \Lambda_2'$ is given by the identification analogous to (1.3),

$$
\mathrm{inv} : G(L)\backslash[G(L)/G_0(\mathcal{O}) \times G(L)/G_0(\mathcal{O})] = G_0(\mathcal{O})\backslash G(L)/G_0(\mathcal{O}) \xrightarrow{\sim} \mathbf{Z}^2 \rtimes S_2. \tag{1.5}
$$

Here $G_0(\mathcal{O})$ denotes the standard Iwahori subgroup of $G(\mathcal{O})$ and on the right appears the extended affine Weyl group $\widetilde{W}$ of GL_2. It is now a pleasant exercise in the Bruhat-Tits building of PGL_2 to see that

$$
\{p\Lambda_2 \subsetneqq \Lambda_1 \subsetneqq \Lambda_2,\ p\Lambda_2' \subsetneqq \Lambda_1' \subsetneqq \Lambda_2';\ p\Lambda_1 \subsetneqq \Lambda_1' \subsetneqq \Lambda_1,\ p\Lambda_2 \subsetneqq \Lambda_2' \subsetneqq \Lambda_2\} \\
= \{(g, g') \in (G(L)/G_0(\mathcal{O}))^2;\ \mathrm{inv}(g, g') \in \mathrm{Adm}(\mu)\}. \tag{1.6}
$$

Here $\mathrm{Adm}(\mu)$ is the following subset of $\widetilde{W}$,

$$
\mathrm{Adm}(\mu) = \{t_{(1,0)}, t_{(0,1)}, t_{(1,0)} \cdot s\}. \tag{1.7}
$$

Here $t_{(1,0)}$ and $t_{(0,1)}$ denote the translation elements in $\widetilde{W} = \mathbf{Z}^2 \rtimes S_2$ corresponding to (1,0) resp. (0,1) in $\mathbf{Z}^2$, and s denotes the non-trivial element in S_2.

For $w \in \widetilde{W}$ and any σ-linear automorphism F of N, let us introduce the *affine Deligne-Lusztig variety*,

$$
X_w(F) = \{g \in G(L)/G_0(\mathcal{O});\ \mathrm{inv}(g, Fg) = w\}. \tag{1.8}
$$

Then we may rewrite (1.4) in the following form, where F_φ denotes the crystalline Frobenius associated to φ,

$$
X(\varphi)_{K_p^{(ii)}} = \bigcup_{w \in \mathrm{Adm}(\mu)} X_w(F_\varphi). \tag{1.9}
$$