Astérisque

SCOTT SHEFFIELD

Random surfaces

Astérisque, tome 304 (2005)

<http://www.numdam.org/item?id=AST_2005__304__R1_0>

ASTÉRISQUE 304

RANDOM SURFACES

Scott Sheffield

Société Mathématique de France 2005
Publié avec le concours du Centre National de la Recherche Scientifique

Scott Sheffield
Courant Institute of Mathematical Sciences, New York University,
251 Mercer Street, New York, NY 10012-1185.
E-mail : sheff@math.nyu.edu

2000 *Mathematics Subject Classification.* — 60D05, 60F10, 60G60.

***Key words and phrases.* —** Gibbs measure, gradient Gibbs measure, random surface, height function, solid-on-solid, localization, crystal facet, rough phase, smooth phase, variational principle, dimer, square ice, Ginzburg-Landau, ferromagnetic, nearest neighbor potential, surface shape, surface tension, specific free energy, Wulff crystal, cluster swapping.

This work is a revised version of a Ph.D. thesis advised by Amir Dembo, whom I thank for tremendous assistance. We spent many hours – sometimes entire days – fleshing out ideas, wading through chapter drafts, and talking through conceptual challenges. He is devoted to his students, has a wonderful sense of humor, and is truly a pleasure to work with.

Thanks also to Persi Diaconis, Michael Harrison, Jon Mattingly, Rafe Mazzeo, and George Papanicolaou for service on oral exam, defense, and reading committees. Particular thanks to Persi Diaconis for extensive advice about the problems in this text and for supervising my first readings in Gibbs measures and statistical physics.

For summer and post-doctoral support and many valuable discussions, I thank the members of the Theory Group at Microsoft Research, particularly Henry Cohn, David Wilson, Oded Schramm, Jennifer Chayes, and Christian Borgs.

Thanks to Richard Kenyon and Andrei Okounkov for their advice and collaboration on the subject of random surfaces derived from perfect matchings in the plane (the *dimer model*). Although the work with Kenyon and Okounkov cites my thesis, the two projects were actually completed in tandem, and the dimer model intuition inspired many of the results presented here.

Thanks also to Jean-Dominique Deuschel for his course on Ginzurg-Landau $\nabla\phi$-interface models at Stanford, and to both Stefan Adams and Jean-Dominique Deuschel for their kind hospitality and extensive, extremely helpful discussions during my visit to Berlin.

Thanks to Jamal Najim for finding and introducing me to Cianchi's modern treatment of Orlicz-Sobolev spaces. Applying these beautiful results to random surfaces was truly a pleasure.

Thanks to Alan Hammond for many relevant conversations, and to a kind friend, Michael Shirts, for managing my thesis submission at Stanford while I was in Paris.

Finally, thanks to my family, especially my wife, Julie, for unwavering patience, love, and support.

RANDOM SURFACES

Scott Sheffield

***Abstract.* —** We study the statistical physical properties of (discretized) "random surfaces," which are random functions from $\mathbb{Z}^d$ (or large subsets of $\mathbb{Z}^d$) to E, where E is $\mathbb{Z}$ or $\mathbb{R}$. Their laws are determined by convex, nearest-neighbor, gradient Gibbs potentials that are invariant under translation by a full-rank sublattice $\mathcal{L}$ of $\mathbb{Z}^d$; they include many discrete and continuous height function models (e.g., domino tilings, square ice, the harmonic crystal, the Ginzburg-Landau $\nabla\phi$ interface model, the linear solid-on-solid model) as special cases.

We prove a *variational principle* – characterizing gradient phases of a given slope as minimizers of the specific free energy – and an empirical measure large deviations principle (with a unique rate function minimizer) for random surfaces on mesh approximations of bounded domains. We also prove that the surface tension is strictly convex and that if u is in the interior of the space of finite-surface-tension slopes, then there exists a minimal energy gradient phase μ_u of slope u.

Using a new geometric technique called *cluster swapping* (a variant of the Swendsen-Wang update for Fortuin-Kasteleyn clusters), we show that μ_u is unique if at least one of the following holds: $E = \mathbb{R}$, $d \in \{1, 2\}$, there exists a *rough gradient phase* of slope u, or u is irrational. When $d = 2$ and $E = \mathbb{Z}$, we show that the slopes of all *smooth phases* (a.k.a. *crystal facets*) lie in the dual lattice of $\mathcal{L}$.

In the case $E = \mathbb{Z}$ and $d = 2$, our results resolve and greatly generalize a number of conjectures of Cohn, Elkies, and Propp – one of which is that there is a unique ergodic Gibbs measure on domino tilings for each non-extremal slope. We also prove several theorems cited by Kenyon, Okounkov, and Sheffield in their recent exact solution of the dimer model on general planar lattices. In the case $E = \mathbb{R}$, our results generalize and extend many of the results in the literature on Ginzurg-Landau $\nabla\phi$-interface models.

***Résumé* (Surfaces aléatoires). —** Nous étudions les propriétés à grande échelle de "surfaces aléatoires" qui sont ici des applications aléatoires de $\mathbb{Z}^d$ (ou de grandes parties de $\mathbb{Z}^d$) à valeurs dans un ensemble E qui est égal à $\mathbb{R}$ ou $\mathbb{Z}$. Leur loi est donnée via un potentiel Gibbsien qui est une fonction convexe des gradients discrets (locaux) de la fonction, et qui est supposé invariant par rapport aux translations d'un véritable sous-réseau $\mathfrak{L}$ de $\mathbb{Z}^d$. Ceci inclut beaucoup de modèles dits de hauteurs (pavages par dominos, glace carrée, le cristal harmonique, le modèle de Ginzburg-Landau $\nabla\phi$, le modèle SOS linéaire).

Nous établissons un principe variationnel qui caractérise les mesures de Gibbs d'une pente donnée comme minimiseurs de l'énergie libre spécifique, et un principe de grande déviations pour la mesure empirique pour les surfaces aléatoires sur des approximations par un réseau de domaines bornés. Nous montrons également que la tension de surface est strictement convexe et que lorsque la pente u définit une tension de surface finie, alors il existe une unique mesure de Gibbs μ_u de pente u ergodique par rapport à $\mathbb{L}$ et d'énergie minimale.

En utilisant une nouvelle idée géométrique de changement de clusters (qui est une variante de l'algorithme de Swendsen-Wang pour les modèles de percolation de Fortuin et Kasteleyn), nous montrons que la mesure μ_u est unique dès lors que l'une des conditions suivantes est vérifiée : $E = \mathbb{R}$, $d = \{1, 2\}$, il existe une mesure de Gibbs "irrégulière" de pente u, ou u est irrationnel.

Lorsque $d = 2$ et $E = \mathbb{Z}$, nous montrons que les pentes de toute mesure lisse (les faces du cristal) sont dans le réseau dual à $\mathfrak{L}$. Lorsque $d = 2$ et $E = \mathbb{Z}$, nos résultats résolvent des conjectures de Cohn, Elkies et Propp ; nous montrons par exemple qu'il existe une seule mesure de Gibbs ergodique sur les pavages par dominos pour chaque pente non-extrémale. Nous établissons aussi des résultats utilisés par Kenyon, Okounkov et Sheffield pour résoudre de manière exacte le modèle de dimères sur des réseaux plans généraux. Lorsque $E = \mathbb{R}$, nos résultats généralisent nombre de résultats sur les modèles d'interfaces $\nabla\phi$ de Ginzburg-Landau.

CONTENTS

CHAPTER 1

INTRODUCTION

The following is a fundamental problem of variational calculus: given a bounded open subset D of $\mathbb{R}^d$ and a free energy function $\sigma : D \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \mapsto \mathbb{R}$, find the differentiable function $f : D \mapsto \mathbb{R}^m$ that (possibly subject to boundary conditions) minimizes the free energy integral:

$$\int_D \sigma(x, f(x), \nabla f(x))\, dx.$$

Since the seventeenth century, these free-energy-minimizing functions have been popular models for determining (among other things) the shapes assumed by solid objects in the presence of outside forces: ropes suspended between poles, elastic sheets stretched to boundary conditions, and twisted or otherwise strained three-dimensional solids. They are also useful in modeling surfaces of water droplets and other phase interfaces. Rigorous formulations and solutions to these problems rank among the great achievements of classical analysis (including work by Fermat, Newton, Leibniz, the Bernoullis, Euler, Lagrange, Legendre, Jacobi, Hamilton, Weierstrass, etc. [**49**]) and play important roles in physics and engineering.

All of these models assume that matter is continuous and distributes force in a continuous way. One of the goals of statistical physics has become not merely to solve variational problems but to understand and, in some sense, to *justify* them in light of the fact that matter is comprised of individual, randomly behaving atoms. To this end, one begins by postulating a simple form for the local particle interactions: one approach – the one we will study in this work – is to represent the "atoms" of the solid crystal by points in a subset Λ of $\mathbb{Z}^d$, each of which has a "spatial position" given by a function $\phi : \Lambda \mapsto \mathbb{R}^m$, and to specify the interaction between the particles by a *Gibbs potential* Φ that possesses certain natural symmetries. The next step is to show that – at least in some "thermodynamic limit" – a random Gibbs configuration will approximate a free-energy-minimizing function like the ones described above.

Another problem, which has no analog in the deterministic, non-atomic classical theory, is the investigation of local statistics of a physical system. How likely are

particular microscopic configurations of atoms to occur as sub-configurations of a larger system? How are these occurrences distributed? To what extent is matter *homogenous* throughout small but non-microscopic regions? Our solutions to these problems will involve *large deviations principles*, which we precisely define later on.

Finally, we want to investigate more directly the connections between the Gibbs potential Φ and the kinds of behavior that can occur in these small but non-microscopic regions. This will require us to ask, given Φ, what are the "gradient phases" (i.e., the ergodic gradient Gibbs measures with finite specific energy) μ of a given slope? Does the μ-variance of the height difference of points n units apart remain bounded independently of n or does it tend to infinity with n? When is the *surface tension* function σ (defined precisely in Chapter 4) strictly convex?

Before we state our results precisely and describe some of the previous work in this area, we will need several definitions. While we attempt to make our exposition relatively self-contained – and define the terms we use precisely – we will also draw heavily from the results in some standard texts: *Sobolev Spaces* by Adams [**1**] and recent extensions by Cianchi ([**14**], [**15**], [**16**]); *Large Deviations Techniques and Applications* by Dembo and Zeitouni [**21**]; *Large Deviations* by Deuschel and Stroock [**25**]; and *Gibbs Measures and Phase Transitions* by Georgii [**42**]. We will carefully state, if not prove, the outside theorems we use.

1.1. Random surfaces and gradient Gibbs measures

1.1.1. Gradient potentials. — The study of random functions ϕ from the lattice $\mathbb{Z}^d$ to a measure space $(E, \mathcal{E})$ is a central component of ergodic theory and statistical physics. In many classical models from physics (e.g., the Ising model, the Potts model, Shlosman's plane rotor model), E is a space with a finite underlying measure λ, $\mathcal{E}$ is the Borel σ-field of E, and $\phi(x)$ has a physical interpretation as the *spin* (or some other internal property) of a particle at location x in a crystal lattice. (See e.g., [**42**].) In the models of interest to us, $(E, \mathcal{E})$ is a space with an infinite underlying measure λ – either $\mathbb{R}^m$ with Lebesgue measure or $\mathbb{Z}^m$ with counting measure – where $\mathcal{E}$ is the Borel σ-algebra of E and $\phi(x)$ usually has a physical interpretation as the *spatial position* of a particle (or the vertical height of a phase interface) at location x in a lattice. For example, if $m = d = 3$, ϕ could describe the spatial positions of the components of an elastic crystal; if $m = 1$ and $d = 2$, ϕ could describe the solid-on-solid or Ginzburg-Landau approximations of a phase interface [**38**].

Throughout the exposition, we denote by Ω the set of functions from $\mathbb{Z}^d$ to E and by $\mathcal{F}$ the Borel σ-algebra of the product topology on Ω. If $\Lambda \subset \mathbb{Z}^d$, we denote by $\mathcal{F}_\Lambda$ the smallest σ-algebra with respect to which $\phi(x)$ is measurable for all $x \in \Lambda$. We write $\mathcal{T}_\Lambda = \mathcal{F}_{\mathbb{Z}^d - \Lambda}$. We write $\Lambda \subset\subset \mathbb{Z}^d$ if Λ is a finite subset of $\mathbb{Z}^d$. A subset of Ω is called a *cylinder set* if it belongs to $\mathcal{F}_\Lambda$ for some $\Lambda \subset\subset \mathbb{Z}^d$. Let $\mathcal{F}$ be the smallest

σ-algebra on Ω containing the cylinder sets. We write $\mathcal{T}$ for the intersection of $\mathcal{T}_\Lambda$ over all finite subsets Λ of $\mathbb{Z}^d$; the sets in $\mathcal{T}$ are called *tail-measurable sets.*

We will also always assume that we are given a family Φ of measurable *potential functions* $\Phi_\Lambda : \Omega \mapsto \mathbb{R} \cup \{\infty\}$ (one for each finite subset Λ of $\mathbb{Z}^d$); each Φ_Λ is $\mathcal{F}_\Lambda$ measurable. We will further assume that Φ is invariant under the group Θ of translations of $\mathbb{Z}^d$ by members of some rank-d lattice $\mathcal{L}$ – i.e., if $s \in \mathcal{L}$, then $\Phi_{\Lambda+s}(\phi_s) = \Phi_\Lambda(\phi)$, where ϕ_s is defined by $\phi_s(i) = \phi(i-s)$. (In many applications, we can take $\mathcal{L} = \mathbb{Z}^d$.) We also assume that Φ is invariant under a group τ of measure-preserving translations of E – i.e., $\Phi_\Lambda(\phi) = \Phi_\Lambda(\tau\phi)$, where $\tau\phi$ is simply defined by $(\tau\phi)(x) = \tau(\phi(x))$. Potentials Φ satisfying the above requirements are called $\Theta \times \tau$*-invariant potentials* or $\mathcal{L} \times \tau$*-invariant potentials.* For all of our main results, we will assume that τ is the full group of translations of $\mathbb{Z}^m$ or $\mathbb{R}^m$; in this case, each $\Phi_\Lambda(\phi)$ is a function of the *gradient* of ϕ, written $\nabla\phi$ and defined by

$$\nabla\phi(x) = (\phi(x+e_1) - \phi(x), \phi(x+e_2) - \phi(x), \ldots, \phi(x+e_d) - \phi(x)),$$

where e_i are the standard basis vectors of $\mathbb{Z}^d$. In this setting, we will refer to $\mathcal{L} \times \tau$-invariant potentials as $\mathcal{L}$*-periodic* or $\mathcal{L}$*-invariant* gradient potentials. We use the term *shift-invariant* to mean $\mathcal{L}$-invariant when $\mathcal{L} = \mathbb{Z}^d$. In some of our applications, we also restrict our attention to *nearest-neighbor potentials*, i.e., those potentials Φ for which $\Phi_\Lambda = 0$ unless Λ is a single pair of adjacent vertices in $\mathbb{Z}^d$. We say that Φ has *finite range* if there exists an r such that $\Phi_\Lambda = 0$ whenever the diameter of Λ is greater than r. For each finite subset Λ of $\mathbb{Z}^d$ we also define a *Hamiltonian*: $H_\Lambda(\phi) = \sum_{\Delta \cup \Lambda \neq \varnothing} \Phi_\Delta(\phi)$, where the sum is taken over finite subsets Δ of $\mathbb{Z}^d$.

We define the *interior Hamiltonian* of Λ, written $H^o_\Lambda(\phi)$, to be:

$$H^o_\Lambda(\phi) = \sum_{\Delta \subset \Lambda} \Phi_\Delta(\phi).$$

This is different from H_Λ because the former sum includes sets Δ that intersect Λ but are not strictly contained in Λ. On the other hand, H^o_Λ is $\mathcal{F}^\tau_\Lambda$ measurable, which is not true of H_Λ. (This H^o_Λ is sometimes called the *free boundary* Hamiltonian for Λ.)

1.1.2. Gibbs Measures. — To define Gibbs measures and gradient Gibbs measures, we will need some additional notation. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be general measure spaces. A function $\pi : \mathcal{X} \times Y \mapsto [0, \infty]$ is called a *probability kernel* from $(Y, \mathcal{Y})$ to $(X, \mathcal{X})$ if

1. $\pi(\cdot|y)$ is a probability measure on $(X, \mathcal{X})$ for each fixed $y \in Y$, and
2. $\pi(A|\cdot)$ is $\mathcal{Y}$-measurable for each fixed $A \in \mathcal{X}$.

Since a probability kernel maps each point in Y to a probability measure on X, we may interpret a probability kernel as giving the law for a random transition from