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A NEW MODEL OF DG-CATEGORIES

by Elena Dimitriadis Bermejo

Abstract. — In this article, we develop a new model for dg-categories. Following Rezk's example in the case of classic Segal spaces, we define dg-Segal spaces as functors between free dg-categories of finite type and simplicial spaces to which we add certain properties. We define also complete dg-Segal spaces and make their relationship to classic Segal spaces explicit. With the help of two new hypercover constructions, and up to a certain hypothesis, we prove that there exists an equivalence between the homotopy category of dg-categories and the homotopy category of functors defined above with a model structure making the complete dg-Segal spaces into its fibrant objects.

Résumé (*Un nouveau modèle des dg-catégories*). — Dans cet article, on développe un nouveau modèle de la catégorie des dg-catégories. En suivant l'exemple de Rezk dans le cas des espaces de Segal classiques, on définit des espaces de dg-Segal: c'est-à-dire, des foncteurs entre les dg-catégories libres de type fini et les espaces simpliciaux, auxquels on ajoute certaines conditions. On définit aussi des espaces de dg-Segal complets, et on explicite leur relation avec les espaces de Segal classiques. Avec l'aide de deux nouvelles constructions d'hyperrecouvrement, et en acceptant une certaine hypothèse, on prouve qu'il existe une équivalence entre la catégorie homotopique des dg-catégories et la catégorie homotopique des foncteurs définis auparavant, avec une structure de modèles qui fait des espaces de dg-Segal complets ses objets fibrants.

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1. Introduction

1.1. A historical perspective. — Dg-categories (i.e. categories enriched over cochain complexes) have been extremely useful over the years in algebraic geometry. Indeed, although classical categories are enough for algebraic topology, algebraic geometry deals with a notion of linearity that does not gel well with those. As such, authors in this field have turned to dg-categories as their generalisation of derived categories. But as useful as these are, they are not without issues. In particular, one glaring problem with dg-categories is the fact that dg-categories have a model structure and a monoidal structure, but the two of them are not compatible.

People working on dg-categories are not the first to encounter that problem, though. Indeed, authors on ∞-categories found themselves dealing with a similar issue in the 1980s: simplicial categories. One of the first models for those, also have a model structure that is not compatible with its monoidal structure. In order to solve that, other models were suggested over the years, each with their own advantages and disadvantages, and were proven to be all Quillen equivalent. Four models, in particular, are the most useful and the most used: simplicial categories, quasi-categories, Segal categories and complete Segal spaces. In recent years, authors have turned repeatedly to those for inspiration for new models for dg-categories.

In 2013, Cohn proved in [5] that there is an equivalence between the underlying ∞-category of the model category of dg-categories localised at the Morita equivalences, and the ∞-category of small idempotent-complete k-linear stable quasi-categories. In 2015, Gepner and Haugseng defined in [9] quasi-categories enriched over a 'nice' Cartesian category V, and a particular case of Haugseng's results states that dg-categories are equivalent (as a quasi-category) to quasi-categories enriched in the derived quasi-category of abelian groups. In another attempt in the same direction, Mertens constructed in 2022 a version of enriched quasi-categories and used it to construct a linear version of the classical dg-nerve functor (see [15] and [16]). In this case, the model structure and the proof of their being Quillen equivalent to dg-categories are still in progress. On the Segal category side of things, in 2020 Bacard defined in [1] a concept of enriched Segal categories, but as with Mertens, the model structure and the equivalence seem to be a future project.

Simplicial categories	dg-categories
quasi-categories [4]	enriched quasi-categories [9] dg-quasi-categories [15]
Segal categories [8]	Segal dg-categories [1]
Complete Segal spaces [18]	??

However, there is no enriched version of complete Segal spaces, and also none of the 'models' that we presented here have been proven to be Quillen equivalent to dg-categories yet. These are the issues we intend to tackle in this article; indeed, we will define a concept of complete dg-Segal space and prove that the model category of complete dg-Segal spaces is Quillen equivalent to that of dg-categories with Tabuada's model structure. Unfortunately, that proof is conditional to a certain result (hypothesis 5.1) that we have been unable to solve for now. We hope to make progress on this in future work.

1.2. Motivation. — Why complete dg-Segal spaces, though? There are two main reasons.

On the one hand, as in the case of complete Segal spaces, our definition of complete dg-Segal spaces will be done as a subcategory of a category of functors, in this case, simplicial functors from free dg-categories of finite type into simplicial sets. It is well known that categories of functors are generally relatively well behaved; for example, the model structure that is so complicated to find in Mertens or Bacard's case will be deduced in a semi-straightforward manner from the one on simplicial sets in our case. But that is not all. To quote Dugger in [6], a subcategory of a diagram category is 'a kind of presentation by generators and relations', and having such a presentation makes the task of defining functors from dg-categories much easier. Indeed, instead of needing to define it over all dg-categories, it would now suffice to define it over free dg-categories of finite type and making sure the 'relations' are sent to weak equivalences.

This approach to complete Segal spaces as a kind of presentation by generators and relations is not new; indeed, in 2005 Toën used the complete Segal space model for ∞-categories in [21] to prove that the group of automorphisms on ∞-categories was $\mathbb{Z}/2\mathbb{Z}$. This result would be generalised in 2021 by Barwick and Schommer–Pries in [2], where they proved that the group of automorphisms on (∞, n)-categories is $(\mathbb{Z}/2\mathbb{Z})^n$ for all $n \in \mathbb{N}^*$. In consequence, it is reasonable to expect that the construction of dg-categories as complete dg-Segal spaces would make it easier to compute the automorphisms of dg-categories.

On the other hand, in the last section we have pointed to the fact that dg-categories have a model structure and a monoidal structure, but that the two of them are not compatible. Indeed, the object $\Delta_k(1,0,1)$ (i.e. the dg-category with two objects and k as the complex of morphisms between the two) is cofibrant in the model category of dg-categories, but it is easy to prove that $\Delta_k(1,0,1) \otimes \Delta_k(1,0,1)$ is not. This is a significant hurdle in the use of dg-categories, as important classic category theory theorems, like the Barr–Beck theorem, can not only not be proven, but cannot even be stated properly in the linear case. Unfortunately, our first and most obvious attempt to find a product that is compatible with the model structure in the complete dg-Segal case, the convolution product, gets thwarted by the fact that free dg-categories are not closed for the product of dg-categories; but there is still reason to believe that finding such a product will be easier in the context of complete dg-Segal spaces, as once again functor categories tend to be better behaved.

1.3. Main results and definitions. — Let k be a commutative ring. We denote by $\mathbf{C}(\mathbf{k})$ the category of cochain complexes, by $c\mathcal{L}$ the category of cofibrant free dg-categories of finite type, and by $c\mathcal{L}_{\mathbb{S}}$ the full subcategory of the simplicial localisation of dg-categories by the weak equivalences that consists only of the objects in $c\mathcal{L}$. Then, in Theorem 3.4 we can define a chain of Quillen adjunctions of the form

$$\mathrm{Re} : \mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet}) \rightleftharpoons \cdots \rightleftharpoons \textbf{dg-cat} : \mathrm{Sing},$$

that can be derived into a single adjunction

$$\mathrm{Ho}(\mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})) \rightleftharpoons \mathrm{Ho}(\textbf{dg-cat}).$$

Our main objective in this paper will be to find the subcategory of $\mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$ such that Sing is an equivalence; that subcategory will be given by our definition of a complete dg-Segal space. In order to get that one, though, we will first need to define dg-Segal spaces. We denote by $Gr(\mathbf{C}(\mathbf{k}))$ the category of graphs on cochain complexes, i.e. a graph whose edges are cochain complexes.

Definition 1.1 (3.8). — Let $F \in \mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$ be a simplicial functor from cofibrant free dg-categories of finite type to simplicial sets. We say that F satisfies the **dg-Segal conditions** if:

1. For all $L, K \in c\mathcal{L}_{\mathbb{S}}$, $F(L \coprod K) \to F(L) \times F(K)$ is a weak equivalence.
2. The image of the initial object is a point, i.e. $F(\emptyset) \simeq *$.
3. Let G be a graph in $Gr(\mathbf{C}(\mathbf{k}))$ and $x, y \in \mathrm{Obj}(G)$. For all $\alpha \in Z^n(G(x,y))$, the image of the free dg-category issued from $G[\not\alpha]$ is a homotopy pullback in $\mathbf{sSet}$ of the following form:

$$\begin{array}{ccc} F(L(G[\not\alpha])) & \longrightarrow & F(L(G)) \\ \big\downarrow & \ulcorner h & \big\downarrow \\ F(\Delta_k^c(1,n,1)) & \longrightarrow & F(\Delta_k(1,n,1)) \end{array}$$

where L is the free functor from graphs to dg-categories, $\Delta_k^c(1,n,1) = L(k^c[n])$ and $\Delta_k(1,n,1) = L(k[n])$.

We denote the full subcategory of $F \in \mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$ that satisfies the dg-Segal conditions by $\mathbf{dg-Segal}$ and call its objects **dg-Segal spaces**.

But we cannot understand that definition on its own; what is $G[\not\alpha]$? Intuitively, if G is a graph over cochain complexes and α is a cocycle in degree n, $G[\not\alpha]$ is a graph with the same objects and the same morphisms, except that we add an element β such that $d(\beta) = \alpha$.

Definition 1.2 (3.6). — Let $G \in Gr(\mathbf{C}(\mathbf{k}))$ be a graph on the category of complexes, $x, y \in \mathrm{Obj}(G)$ two objects in G, and $\alpha \in Z^n(G(x,y))$ a cycle in $G(x,y)$. We define the graph $G[\not\alpha]$ to be a pushout in the category of graphs over the morphism $k[n] \to k^c[n]$, where $k[n]$ is the graph with two objects and the morphism complex between the two is a complex concentrated in degree n, where it is k; and $k^c[n]$ is the graph with two objects and the complex between the two that is always zero except for the degrees $n-1$ and n, where it is k.

$$\begin{array}{ccc} k[n] & \xrightarrow{\alpha} & G \\ \downarrow & & \downarrow \\ k^c[n] & \longrightarrow & G[\not\alpha]. \end{array}$$

We can prove that every object in the image of Sing is a dg-Segal space; and we can also construct a model structure for $\mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$ where the dg-Segal spaces are its fibrant objects. But as happened in the case of classical Segal spaces and ∞-categories, this object is not enough to completely determine dg-categories. As such, we will define complete dg-Segal spaces. For those, we will use the classical definition of a complete Segal space.

Proposition 1.3 (3.13). — *There exists a morphism, called* ***the linearisation of*** Δ*, between the categories* Δ *and* $c\mathcal{L}_{\mathbb{S}}$*, and it defines a Quillen adjunction between the categories* $\mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$ *and* $\mathrm{Fun}(\Delta^{op}, \mathbf{sSet})$ *with their respective projective structures.*

We call the morphism $j^* : \mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet}) \to \mathrm{Fun}^{\mathbb{S}}(\Delta^{op}, \mathbf{sSet})$ *the* ***delinearisation morphism****.*

As we now have a direct way of comparing dg-Segal spaces and classical Segal spaces, we can prove that the image by the delinearisation functor of a dg-Segal space is always a Segal space; using that, we will define the complete dg-Segal spaces as being the dg-Segal spaces such that their delinearised version is a complete Segal space.

Definition 1.4 (3.27). — Let F be an object in $\mathrm{Fun}^{\mathbb{S}}(c\mathcal{L}_{\mathbb{S}}^{op}, \mathbf{sSet})$. We say that F is **a complete dg-Segal space** if it is a dg-Segal space, and the morphism

$$F(k) \to F_{hoequiv}$$

is a weak equivalence, where $F_{hoequiv}$ is the subset of $F(\Delta_k^1)$ whose 0-simplexes are homotopy equivalences.

As in the case of dg-Segal spaces, we can prove that the image of Sing is also contained in complete dg-Segal spaces, and also we can define a model structure such that the complete dg-Segal spaces are its fibrant objects.