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# THE SCHUR MULTIPLIER OF FINITE SYMPLECTIC GROUPS

BY LOUIS FUNAR & WOLFGANG PITSCH

ABSTRACT. — We show that the Schur multiplier of $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ is $\mathbb{Z}/2\mathbb{Z}$, when $D$ is divisible by 4.

RÉSUMÉ (*Multiplicateur de Schur des groupes symplectiques finis*). — Nous montrons que le multiplicateur de Schur de $Sp(2g, \mathbb{Z}/D\mathbb{Z})$ est $\mathbb{Z}/2\mathbb{Z}$ quand $D$ est divisible par 4.

## 1. Introduction and statements

Let $g \geq 1$ be an integer and denote by $Sp(2g, \mathbb{Z})$ the symplectic group of $2g \times 2g$ matrices with integer coefficients. Deligne's nonresidual finiteness theorem from [5] states that the *universal central extension* $\widetilde{Sp(2g, \mathbb{Z})}$ is not residually finite, since the image of its center under any homomorphism into a finite group has an order of at most 2 when $g \geq 3$. Our first motivation was to understand this result and give a sharp statement, namely to decide whether the

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image of the central $\mathbb{Z}$ might be of order 2. Since these symplectic groups have the congruence subgroup property, this boils down to understanding the second homology of symplectic groups with coefficients in finite cyclic groups. In the sequel, for simplicity and unless otherwise explicitly stated, all (co)homology groups will be understood to be with trivial integer coefficients. An old theorem of Stein (see [14], Thm. 2.13 and Prop. 3.3.a) is that $H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = 0$, when $D$ is not divisible by 4. The case $D \equiv 0 \pmod 4$ has remained open since then; this is explicitly mentioned, for instance, in [13] (Remarks after Thm. 3.8). Our main result settles this case.

Theorem 1.1. — *The second homology group of finite principal congruence quotients of $Sp(2g,\mathbb{Z})$, $g \geq 3$ is*

$$H_2(Sp(2g,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}, \text{ if } D \equiv 0 \pmod 4.$$

In comparison, recall that Beyl (see [2]) showed that $H_2(SL(2,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0 \pmod 4$, and Dennis and Stein proved using K-theoretic methods that for $n \geq 3$, we have $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}$, for $D \equiv 0 \pmod 4$, while $H_2(SL(n,\mathbb{Z}/D\mathbb{Z})) = 0$, for $D \not\equiv 0 \pmod 4$ (see [6], Cor. 10.2 and [11], Section 12).

Our proof also relies on Deligne's nonresidual finiteness theorem from [5] and deep results of Putman in [13], and shows that we can detect this $\mathbb{Z}/2\mathbb{Z}$ factor on $H_2(Sp(2g,\mathbb{Z}/32\mathbb{Z}))$ for $g \geq 4$, providing an explicit extension that detects this homology class.

## 2. Preliminaries

**2.1. Residual finiteness of universal central extensions**. — In this section, we collect results about universal central extensions of perfect groups, for the sake of completeness of our arguments. Every perfect group $\Gamma$ has a universal central extension $\widetilde{\Gamma}$; the kernel of the canonical projection map $\widetilde{\Gamma} \to \Gamma$ contains the center $Z(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$ and is canonically isomorphic to the second integral homology group $H_2(\Gamma)$. We will recall now how the residual finiteness problem for the universal central $\widetilde{\Gamma}$ of a perfect and residually finite group $\Gamma$ translates into an homological problem about $H_2(\Gamma)$. We start with a classical result for maps between universal central extensions of perfect groups.

Lemma 2.1. — *Let $\Gamma$ and $F$ be perfect groups, $\widetilde{\Gamma}$ and $\widetilde{F}$ be their universal central extensions and $p : \Gamma \to F$ be a group homomorphism. Then, there exists a unique homomorphism $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$ lifting $p$ such that the following* *diagram is commutative:*

$$\begin{array}{ccccccc} 1 \to H_2(\Gamma) & \to & \widetilde{\Gamma} & \to & \Gamma & \to 1 \\ p_* \downarrow & & \widetilde{p}\downarrow & & \downarrow p & \\ 1 \to H_2(F) & \to & \widetilde{F} & \to & F & \to 1 \end{array}$$

For a proof we refer the interested reader to ([10], Chap. VIII) or ([4], Chap. IV, Ex. 1, 7). If $\Gamma$ is a perfect residually finite group, to prove that its universal central extension $\widetilde{\Gamma}$ is also residually finite we only have to find enough finite quotients of $\widetilde{\Gamma}$ to detect the elements in its center $H_2(\Gamma)$. The following lemma analyzes the situation.

LEMMA 2.2. — *Let $\Gamma$ be a perfect group and denote by $\widetilde{\Gamma}$ its universal central extension.*

1. *Let $H$ be a finite index normal subgroup $H \subset \Gamma$ such that the image of $H_2(H)$ into $H_2(\Gamma)$ contains the subgroup $dH_2(\Gamma)$, for some $d \in \mathbb{Z}$. Let $F = \Gamma/H$ be the corresponding finite quotient of $\Gamma$ and $p : \Gamma \to F$ the quotient map. Then, $d \cdot p_*(H_2(\Gamma)) = 0$, where $p_* : H_2(\Gamma) \to H_2(F)$ is the homomorphism induced by $p$. In particular, if $p_* : H_2(\Gamma) \to H_2(F)$ is surjective, then $d \cdot H_2(F) = 0$.*
2. *Assume that $F$ is a finite quotient of $\Gamma$ satisfying $d \cdot p_*(H_2(\Gamma)) = 0$. Let $\widetilde{F}$ denote the universal central extension of $F$. Then, the homomorphism $p : \Gamma \to F$ has a unique lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$, and the kernel of $\widetilde{p}$ contains $d \cdot H_2(\Gamma)$.*

Observe that in point 2. of Lemma 2.2 the group $F$ being finite, $H_2(F)$ is also finite, and, hence, one can take $d = |H_2(F)|$.

*Proof.* — The image of $H$ into $F$ is trivial, and, thus, the image of $H_2(H)$ into $H_2(F)$ is trivial. This implies that $p_*(d \cdot H_2(\Gamma)) = 0$, which proves the first part of the lemma.

Further, by Lemma 2.1 there exists a unique lift $\widetilde{p} : \widetilde{\Gamma} \to \widetilde{F}$. If $d \cdot p_*(H_2(\Gamma)) = 0$, then Lemma 2.1 yields $d \cdot \widetilde{p}(c) = d \cdot p_*(c) = 0$, for any $c \in H_2(\Gamma)$. This settles the second part of the lemma. □

REMARK 2.3. — It might be possible that we have $d' \cdot p_*(H_2(\Gamma)) = 0$ for some proper divisor $d'$ of $d$, so the first part of Lemma 2.2 can only give an upper bound of the orders of the image of the second cohomology. In order to find lower bounds we need additional information concerning the finite quotients $F$.

LEMMA 2.4. — *Let $\Gamma$ be a perfect group, $\widetilde{\Gamma}$ its universal central extension, $p : \Gamma \to F$ a surjective homomorphism onto a finite group $F$, and $\widehat{p} : \widetilde{\Gamma} \to G$ be some lift of $p$ to a central extension $G$ of $F$ by some finite abelian group $C$. Assume that the image of $H_2(\Gamma) \subset \widetilde{\Gamma}$ in $G$ by $\widehat{p}$ contains an element of order $q$. Then there exists an element of $p_*(H_2(\Gamma)) \subset H_2(F)$ of order $q$.*