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A CONJECTURAL BOUND ON THE SECOND BETTI NUMBER FOR HYPER-KÄHLER MANIFOLDS

by Yoon-Joo Kim & Radu Laza

Abstract. — In a previous work ([1]), we noted that the known cases of hyper-Kähler manifolds satisfy a natural condition on the LLV decomposition of the cohomology; informally, the Verbitsky component is the dominant representation in the LLV decomposition. Assuming this condition holds for all hyper-Kähler manifolds, we obtain an upper bound for the second Betti number in terms of the dimension.

Résumé (*Une majoration conjecturale sur le deuxième nombre de Betti pour les variétés hyper-kählériennes*). — Dans un article précédent, nous avons remarqué que les exemples connus de variétés hyper-kählériennes satisfont une condition naturelle sur la décomposition LLV de la cohomologie; informellement, la composante Verbitsky est la représentation dominante dans la décomposition LLV. Supposons que toutes les variétés hyper-kählériennes satisfait cette condition, nous obtenons un majorant pour le deuxième nombre de Betti en fonction de la dimension de la variété.

1. Introduction

A fundamental open question in the theory of compact hyper-Kähler manifolds is the boundedness question: *are there finitely many diffeomorphism types*

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of hyper-Kählers in a given dimension? In accordance with the Torelli principle, Huybrechts [3, Thm 4.3] proved that there are finitely many diffeomorphism types of hyper-Kähler manifolds, once the dimension and the (unnormalized) Beauville–Bogomolov lattice $(H^2(X,\mathbb{Z}), q_X)$ are fixed. Thus, bounding the hyper-Kähler manifolds is equivalent to bounding the second Betti number $b_2 = b_2(X)$, and then the Beauville–Bogomolov form (e.g., the discriminant). In dimension 2, a compact hyper-Kähler manifold is always a K3 surface, thus $b_2 = 22$. In dimension 4, Beauville and Guan [2] gave a sharp bound $b_2 \leq 23$ (in fact, Guan showed that $3 \leq b_2 \leq 8$ or $b_2 = 23$). For some further partial results on bounding b_2, see Remark 1.2. The purpose of this note is to give a conjectural bound on $b_2(X)$ for an arbitrary compact hyper-Kähler manifold X of dimension $2n$. Our bound depends on a natural conjectural condition satisfied by the Looijenga–Lunts–Verbitsky (LLV) decomposition of the cohomology $H^*(X)$ for hyper-Kähler manifolds X.

To state our results, let us recall that Verbitsky [9] and Looijenga–Lunts [6] noted that the cohomology $H^*(X)$ of a hyper-Kähler manifold admits a natural action by the Lie algebra $\mathfrak{g} = \mathfrak{so}(b_2 + 2)$, generalizing the usual hard Lefschetz theorem. As a $\mathfrak{g}$-module, the cohomology of a hyper-Kähler manifold X decomposes as

$$H^*(X) = \bigoplus_{\mu} V_\mu^{\oplus m_\mu}, \tag{1}$$

where V_μ indicates an irreducible $\mathfrak{g}$-module of the highest weight $\mu = (\mu_0, \ldots, \mu_r)$, with $r = \left\lfloor \frac{b_2(X)}{2} \right\rfloor = \operatorname{rk}\mathfrak{g} - 1$. We refer to $\mathfrak{g}$ as the *LLV algebra* of X and to (1) as the *LLV decomposition* of $H^*(X)$ (see [1] for a further discussion). Motivated by the behavior of the LLV decomposition in the known cases of hyper-Kähler manifolds [1], we have made the following conjecture.

CONJECTURE ([1]). — *Let X be a compact hyper-Kähler manifold of dimension $2n$. Then, the weights $\mu = (\mu_0, \ldots, \mu_r)$ occurring in the LLV decomposition* (1) *of $H^*(X)$ satisfy*

$$\mu_0 + \cdots + \mu_{r-1} + |\mu_r| \leq n. \tag{2}$$

The conjecture holds for all currently known examples of compact hyper-Kähler manifolds (cf. [1, §1]). Furthermore, the equality in (2) holds for *the Verbitsky component*, an irreducible $\mathfrak{g}$-submodule with the highest weight $\mu = (n, 0, \ldots, 0)$ that is always present in $H^*(X)$. This shows that (2) is sharp. Beyond the evidence given by the validity of (2) in the known cases, we have some partial arguments of motivic nature (and depending on standard conjectures) showing that at least (2) is plausible. This will be discussed elsewhere. The purpose of this note is to show that conjecture (2) implies a general bound on $b_2(X)$.

Main Theorem. — *Let X be a compact hyper-Kähler manifold of dimension $2n$. If the condition (2) holds for X, then*

$$b_2(X) \le \begin{cases} \frac{21+\sqrt{96n+433}}{2} & \text{if } H^*_{\mathrm{odd}}(X)=0 \\ 2k+1 & \text{if } H^k(X)\neq 0 \text{ for some odd } k \end{cases}. \tag{3}$$

Remark 1.1. — A slightly weaker version of (3) is

$$b_2(X) \le \max\left\{\tfrac{21+\sqrt{96n+433}}{2}, 4n-1\right\},$$

which reads explicitly

n	1	2	3	4	5	6	7	≥ 8
$b_2(X) \le$	22	23	23	24	25	26	27	$4n-1$

In low dimensions, our bounds agree with the known results and seem fairly sharp. For instance, we know $\mathrm{K3}^{[n]}$-type hyper-Kähler manifolds have $b_2=23$, and OG10 manifolds have $b_2=24$. These examples almost reach our maximum bound of b_2 for low dimensions. Similarly, Kum_n type hyper-Kähler manifolds have $H^3(X)\neq 0$ and $b_2=7$, also showing that the second inequality in (3) is sharp.

Remark 1.2. — Sawon [8] and Kurnosov [5] previously obtained the same bounds for $3\le n\le 5$, and also predicted the general formula (3) when $H^*_{\mathrm{odd}}(X)=0$. However, their results were based on the assumption that an irreducible module V_μ is determined by the shape of its Hodge diamond. In general, the shape of the Hodge diamond of V_μ is controlled only by the first two coefficients μ_0,μ_1 (see [1, §2.2]).

A few words about the proof of our conjectural bound. First, in [1, §1], we already obtained that the condition (2) has consequences on the odd cohomology (specifically, if $b_2\ge 4n$, then there should be no odd cohomology). A slight generalization of the argument in [1, §1] then gives the second inequality in (3). The main content of this note is the control of the even cohomology under the assumption (2). Essentially, our argument is a representation theoretic refinement of Beauville's argument that $b_2\le 23$ for hyper-Kähler fourfolds. Namely, the starting point is Salamon's relation [7], a linear relation satisfied by the Betti numbers of hyper-Kähler manifolds. Inspired by the shape of it, we define a numerical function $s(W)$ for a $\mathfrak{g}$-module W and verify its basic properties, most importantly $s(W_1\oplus W_2)\le\max\{s(W_1),s(W_2)\}$. In this setting,