MÉMOIRES DE LA SMF 107

MATHEMATICAL STUDY OF THE BETAPLANE MODEL: EQUATORIAL WAVES AND CONVERGENCE RESULTS

Isabelle Gallagher

Laure Saint-Raymond

Société Mathématique de France 2006

Publié avec le concours du Centre National de la Recherche Scientifique

I. Gallagher
Institut de Mathématiques de Jussieu, Université Paris VII, 175, rue du Chevaleret, 75013 Paris, France.
E-mail : `Isabelle.Gallagher@math.jussieu.fr`

L. Saint-Raymond
Laboratoire J.-L. Lions UMR 7598, Université Paris VI, 175, rue du Chevaleret, 75013 Paris, France.
E-mail : `saintray@ann.jussieu.fr`

2000 *Mathematics Subject Classification*. — 35Q30, 35P99, 76U05, 86A10.

***Key words and phrases*. —** Rotating fluid, betaplane, Kelvin wave, Poincaré wave, Rossby wave, equatorial trapping, weak compactness, filtering method, harmonic analysis.

The authors wish to thank warmly J.-Y. Chemin and F. Golse for their patience in answering their many questions on a number of subjects addressed in this work. They also thank the anonymous referee for pointing out an important improvement in the study of resonances (namely the absence of resonances in Rossby modes).

MATHEMATICAL STUDY OF THE BETAPLANE MODEL: EQUATORIAL WAVES AND CONVERGENCE RESULTS

Isabelle Gallagher, Laure Saint-Raymond

***Abstract*. —** We are interested in a model of rotating fluids, describing the motion of the ocean in the equatorial zone. This model is known as the Saint-Venant, or shallow-water type system, to which a rotation term is added whose amplitude is linear with respect to the latitude; in particular it vanishes at the equator. After a physical introduction to the model, we describe the various waves involved and study in detail the resonances associated to those waves. We then exhibit the formal limit system (as the rotation becomes large), obtained as usual by filtering out the waves, and prove its wellposedness. Finally we prove three types of convergence results: a weak convergence result towards a linear, geostrophic equation, a strong convergence result of the filtered solutions towards the unique strong solution to the limit system, and finally a “hybrid” strong convergence result of the filtered solutions towards a weak solution to the limit system. In particular we obtain that there are no confined equatorial waves in the mean motion as the rotation becomes large.

***Résumé* (Étude mathématique du modèle bétaplan : ondes équatoriales et résultats de convergence)**

On s’intéresse à un modèle de fluides en rotation rapide, décrivant le mouvement de l’océan dans la zone équatoriale. Ce modèle est connu sous le nom de Saint-Venant, ou système « shallow water », auquel on ajoute un terme de rotation dont l’amplitude est linéaire en la latitude ; en particulier il s’annule à l’équateur. Après une introduction physique au modèle, on décrit les différentes ondes en jeu et l’on étudie en détail les résonances associées à ces ondes. On exhibe ensuite un système limite formel (dans la limite d’une forte rotation), obtenu comme d’habitude en filtrant les ondes, et l’on démontre qu’il est bien posé. Enfin on démontre trois types de résultats de convergence : un théorème de convergence faible vers un système géostrophique linéaire, un théorème de convergence forte des solutions filtrées vers la solution unique du système limite, et enfin un résultat « hybride » de convergence forte des solutions filtrées vers une solution faible du système limite. En particulier on démontre l’absence d’ondes équatoriales confinées dans le mouvement moyen, quand la rotation augmente.

CONTENTS

1. Introduction 1
1.1. Physical phenomenon observed in the equatorial zone of the earth 2
1.2. A mathematical model for the ocean in the equatorial zone 3
1.3. Some orders of magnitude in the equatorial zone 4
1.4. The Cauchy problem for the betaplane model 5
1.5. Structure of the paper 7

2. Equatorial waves 9
2.1. The geostrophic constraint 9
2.2. Description of the waves 11
2.3. The filtering operator and the formal limit system 22
2.4. Interactions between equatorial waves 24

3. The envelope equations 33
3.1. Definition of suitable functional spaces 33
3.2. Statement of the wellposedness result 36
3.3. Preliminary results 37
3.4. Proof of Theorem 2 46
3.5. Proof of Theorem 3 57
3.6. A regularity result for the divergence 60

4. Convergence results 65
4.1. Weak convergence of weak solutions 66
4.2. Strong convergence of filtered weak solutions towards a strong solution . 76
4.3. Strong convergence of filtered weak solutions towards a weak solution . . . 90

Bibliography 111

Notation Index 115

CHAPTER 1

INTRODUCTION

The aim of this paper is to obtain a description of geophysical flows, especially oceanic flows, in the equatorial zone. For the scales considered, *i.e.*, on domains extending over many thousands of kilometers, the forces with dominating influence are the gravity and the Coriolis force. The question is therefore to understand how they counterbalance eachother to impose the so-called geostrophic constraint on the mean motion, and to describe the oscillations which are generated around this geostrophic equilibrium.

At mid-latitudes, on "small" geographical zones, the variations of the Coriolis force due to the curvature of the Earth are usually neglected, which leads to a singular perturbation problem with constant coefficients. The corresponding asymptotics, called asymptotics of rotating fluids, have been studied by a number of authors. We refer for instance to the pioneering work [**22**] and to the review by R. Temam and M. Ziane [**34**], or to the work by J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [**4**].

In order to a get a more realistic description, which allows for instance to exhibit the specificity of the equatorial zone, one has to study more intricate models, taking into account especially the interaction between the fluid and the atmosphere (free surface), and the geometry of the Earth (variations of the local vertical component of the Earth rotation). The mathematical modelling of these various phenomenon, as well as their respective importance according to the scales considered, have been studied in a rather systematic way by A. Majda [**25**], and R. Klein and A. Majda [**19**]. We refer also to [**9**] for a review of mathematical methods for the study of geophysical fluids.

Here we will focus on *quasigeostrophic, oceanic* flows, meaning that we will consider horizontal length scales of order 1000km and vertical length scales of order 5 km, so that the aspect ratio is very small and the shallow-water approximation is relevant (see for instance the works by D. Bresch, B. Desjardins and C.K. Lin [**3**] or by J.-F. Gerbeau and B. Perthame [**13**]). In this framework, the asymptotics of homogeneous rotating fluids have been studied by D. Bresch and B. Desjardins [**2**].

For the description of *equatorial* flows, one has to take further into account the variations of the Coriolis force, and especially the fact that it cancels at equator. The inhomogeneity of the Coriolis force has already been studied by B. Desjardins and E. Grenier [**6**] and by the authors [**10**] for an incompressible fluid without pressure, and [**11**] for an incompressible fluid with rigid lid upper boundary (see also [**7**] for a study of the wellposedness and weak asymptotics of a non-viscous model). The question here is then to understand the combination of the effects due to the free surface, and of the effects due to the variations of the Coriolis force.

Note that, for the sake of simplicity, we will not discuss the effects of the interaction with the boundaries, describing neither the vertical boundary layers, known as Ekman layers (see for instance the paper by D. Gérard-Varet [**12**]), nor the lateral boundary layers, known as Munk and Stommel layers (see for instance [**6**]). We will indeed consider a purely horizontal model, assuming periodicity with respect to the longitude (omitting the stopping conditions on the continents) and and infinite domain for the latitude (using the exponential decay of the equatorial waves to neglect the boundary).

1.1. Physical phenomenon observed in the equatorial zone of the earth

The rotation of the earth has a dominating influence on the way the atmosphere and the ocean respond to imposed changes. The dynamic effect is caused (see [**14**], [**16**], [**28**]) by the Coriolis acceleration, which is equal to the product of the Coriolis parameter f and the horizontal velocity.

An important feature of the response of a rotating fluid to gravity is that it does not adjust to a state of rest, but rather to an equilibrium which contains more potential energy than does the rest state. The steady equilibrium solution is a geostrophic balance, *i.e.*, a balance between the Coriolis acceleration and the pressure gradient divided by density. The equation determining this steady solution contains a length scale a, called the Rossby radius of deformation, which is equal to $c/|f|$ where c is the wave speed in the absence of rotation effects. If f tends to zero, then a tends to infinity, indicating that for length scales small compared with a, rotation effects are small, whereas for scales comparable to or larger than a, rotation effects are important. Added to that mean, geostrophic motion, are time oscillations which correspond to the so-called ageostrophic motion. The use of a constant-f approximation to describe motion on the earth is adequate to handle the adjustment process at mid-latitudes: Kelvin [**35**] stated that his wave solutions (also known as Poincaré waves) are applicable *"in any narrow lake or portion of the sea covering not more than a few degrees of the earth's surface, if for $\frac{1}{2}f$ we take the component of the earth's angular velocity* *round a vertical through the locality, that is to say*

$$\frac{1}{2}f = \Omega \sin \phi,$$

where Ω denotes the earth's angular velocity and ϕ the latitude."

The adjustment processes are somewhat special when the Coriolis acceleration vanishes: the equatorial zone is actually found to be a waveguide: as explained in [**14**], there is an equatorial Kelvin wave, and there are equatorially trapped waves, which are the equivalent of the Poincaré waves in a uniformly rotating system. There is also an important new class of waves with much slower frequencies, called planetary or quasi-geostrophic waves. These owe their existence to the variations in the undisturbed potential vorticity and thus exist at all latitudes. However, the ray paths along which they propagate bend, as do the paths of gravity waves, because of the variation of Coriolis parameter with latitude, and it is this bending that tends to confine the waves to the equatorial waveguide.

1.2. A mathematical model for the ocean in the equatorial zone

In order to explore the qualitative features of the equatorial flow, we restrict our attention here to a very simplified model of oceanography. More precisely, we consider the ocean as an incompressible viscous fluid with free surface submitted to gravitation, and further make the following classical assumptions:

$(H1)$ the density of the fluid is homogeneous,

$(H2)$ the pressure law is given by the hydrostatic approximation,

$(H3)$ the motion is essentially horizontal and does not depend on the vertical coordinate,

leading to the so-called shallow water approximation.

We therefore consider a so-called viscous Saint-Venant model, which describes vertically averaged flows in three dimensional shallow domains in terms of the horizontal mean velocity field u and the depth variation h due to the free surface. Taking into account the Coriolis force, a particular model reads as

$$\begin{aligned} \partial_t h + \nabla\cdot (hu) = 0 \\ \partial_t(hu) + \nabla\cdot (hu \otimes u) + f(hu)^\perp + \frac{1}{\mathrm{Fr}^2} h\nabla h - h\nabla K(h) - A(h,u) = 0 \end{aligned} \tag{1.2.1}$$

where f denotes the vertical component of the earth rotation, Fr the Froude number, and K and A are the capillarity and viscosity operators. We have written $u^\perp$ for the vector $(u_2, -u_1)$.

Note that, from a theoretical point of view, it is not clear that the use of the shallow water approximation is relevant in this context since the Coriolis force is known to generate vertical oscillations which are completely neglected in such an approach. Nevertheless, this very simplified model is commonly used by physicists [**14, 29**] and we will see that its study already gives a description of the horizontal motion corresponding to experimental observations.

Of course, in order that the curvature of the earth can be neglected, and that latitude and longitude can be considered as cartesian coordinates, we should consider only a thin strip around the equator. This means that we should study (1.2.1) on a bounded domain, and supplement it with boundary conditions. Nevertheless, as we expect the Coriolis force to confine equatorial waves, we will perform our study on $\mathbf{R} \times \mathbf{T}$ where $\mathbf{T}$ is the one-dimensional torus $\mathbf{R}_{/2\pi\mathbf{Z}}$, and check a posteriori that oscillating modes vanish far from the equator, so that it is reasonable to conjecture that they should not be disturbed by boundary conditions.

1.3. Some orders of magnitude in the equatorial zone

For motions near the equator, the approximations

$$\sin\phi \sim \phi, \quad \cos\phi \sim 1$$

may be used, giving what is called the equatorial betaplane approximation. Half of the earth's surface lies at latitudes of less than 30^o and the maximum percentage error in the above approximation in that range of latitudes is only 14%. In this approximation, f is given by

$$f = \beta x_1,$$

where x_1 is distance northward from the equator, taking values in the range

$$x_1 \in [-3000\,\mathrm{km}\,,\ 3000\,\mathrm{km}],$$

and β is a constant given by

$$\beta = \frac{2\Omega}{r} = 2.3 \times 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}.$$

A formal analysis of the linearized versions of the equations shows then that rotation effects do not allow the motion in each plane $x_1 = const$ to be independent because a geostrophic balance between the eastward velocity and the north-south pressure gradient is required. Equatorial waves actually decay in a distance of order a_e, the so-called equatorial radius of deformation,

$$a_e = \left(\frac{c}{2\beta}\right)^{1/2}$$

where c is the square root of gH, H being interpreted as the equivalent depth. For baroclinic ocean waves, appropriate values of c are typically in the range $0.5\mathrm{ms}^{-1}$ to $3\mathrm{ms}^{-1}$, so the order of the equatorial Rossby radius is

$$a_e \sim 100\,\mathrm{km},$$