Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

HIGHER HOCHSCHILD HOMOLOGY AND EXPONENTIAL FUNCTORS

Geoffrey Powell & Christine Vespa

Tome 153
Fascicule 1

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 1-141

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 153, mars 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (1), 2025, p. 1-141

HIGHER HOCHSCHILD HOMOLOGY AND EXPONENTIAL FUNCTORS

BY GEOFFREY POWELL & CHRISTINE VESPA

ABSTRACT. — We study higher Hochschild homology evaluated on wedges of circles, viewed as a functor on the category of free groups. The main results use coefficients arising from square-zero extensions; this is motivated by work of Turchin and Willwacher in relation to hairy graph cohomology.

The functorial point of view allows us to exploit tools such as the theory of polynomial functors and exponential functors. We also introduce and make essential use of the category of outer functors, the full subcategory of functors on free groups on which inner automorphisms act trivially.

We give a description of higher Hochschild homology in terms of intrinsically defined polynomial outer functors; we also obtain several explicit computations of these outer functors, working over a field of characteristic zero. In particular, higher Hochschild homology gives a natural source of non-trivial polynomial outer functors.

Texte reçu le 11 octobre 2019, modifié le 9 septembre 2024, accepté le 31 octobre 2024.

GEOFFREY POWELL, Univ Angers, CNRS, LAREMA, SFR MATHSTIC, F-49000 Angers, France • *E-mail :* `Geoffrey.Powell@math.cnrs.fr` • *Url :* `https://math.univ-angers.fr/~powell/`

CHRISTINE VESPA, Aix-Marseille Université/Centrale Marseille, Institut de Mathématiques de Marseille, Marseille, France. • *E-mail :* `christine.vespa@univ-amu.fr` • *Url :* `https://www.i2m.univ-amu.fr/perso/christine.vespa/`

Mathematical subject classification (2010). — 55N99, 18A25, 13D03, 20J05.

Key words and phrases. — Functors, outer functors, exponential functors, higher Hochschild homology, free groups, outer automorphisms, Schur functor.

This work was partially supported by the ANR Project *ChroK*, `ANR-16-CE40-0003`. The first author was also supported by the project *Nouvelle Équipe*, convention `2013-10203/10204` between the Région des Pays de la Loire and the Université d'Angers. The second author was also supported by the ANR projects *AlMaRe* `ANR-19-CE40-0001-01`, *HighAGT* `ANR-20-CE40-0016`, and *SHoCoS* `ANR-22-CE40-0008`.

doi:10.24033/bsmf.2897

RÉSUMÉ (*Homologie de Hochschild supérieure et foncteurs exponentiels*). — On étudie l'homologie de Hochschild supérieure évaluée sur des bouquets de cercles en tant que foncteur sur la catégorie des groupes libres. Les résultats principaux utilisent des coefficients provenant des extensions à carré nul. Ceci est motivé par le travail de Turchin et Willwacher en lien avec la cohomologie des graphes chevelus.

Le point de vue fonctoriel nous permet d'exploiter des outils tels que la théorie des foncteurs polynomiaux et celle des foncteurs exponentiels. On introduit et on utilise de manière essentielle la catégorie des outre-foncteurs, qui est la sous-catégorie pleine des foncteurs sur les groupes libres sur lesquels les automorphismes intérieurs agissent trivialement.

Nous donnons une description de l'homologie de Hochschild supérieure en termes d'outre-foncteurs polynomiaux définis intrinsèquement. Nous obtenons également plusieurs calculs explicites de ces outre-foncteurs lorsqu'on travaille sur un corps de caractéristique nulle. En particulier, l'homologie de Hochschild supérieure est une source naturelle d'outre-foncteurs polynomiaux non-triviaux.

1. Introduction

Higher Hochschild homology, defined by Pirashvili [31], is a generalisation of the classical Hochschild homology for commutative rings. It is a (non-additive) homology theory for pointed topological spaces. We denote by $HH_*(X; L)$ the higher Hochschild homology of a pointed space X with coefficients in a Γ-module L, where Γ is the category of finite pointed sets. A fundamental example of a Γ-module is given by the Loday construction $\mathcal{L}(A, M)$ for a commutative algebra A and an A-module M. Taking X to be the circle S^1, $HH_*(S^1; \mathcal{L}(A, M))$ identifies with the classical Hochschild homology $HH_*(A, M)$.

In this paper, we focus on the case where X is a finite wedge of circles $(S^1)^{\vee r}$, for $r \in \mathbb{N}$. This is inspired by the work of Turchin and Willwacher [41] in connection with the calculation of hairy graph homology [40]. The homology $HH_*((S^1)^{\vee r}; \mathcal{L}(A, M))$ is particularly interesting because it can be seen as a natural representation of the group $\mathrm{Aut}(\mathbb{Z}^{\star r})$, where $\mathbb{Z}^{\star r}$ is the free group. Turchin and Willwacher observed in [41] that, when the coefficients are of the specific type $\mathcal{L}(A, A)$, $HH_*((S^1)^{\vee r}; \mathcal{L}(A, A))$ is a representation of the outer automorphism group $\mathrm{Out}(\mathbb{Z}^{\star r})$. Since the representation theory of $\mathrm{Out}(\mathbb{Z}^{\star r})$ is not yet well understood, having concrete examples of such representations is particularly valuable.

Instead of examining these representations individually for a fixed value of r, we introduce a novel approach by studying them through a functorial perspective. Specifically, by interpreting the finite wedge $(S^1)^{\vee r}$ as the classifying space $B\mathbb{Z}^{\star r}$ of the free group $\mathbb{Z}^{\star r}$, we obtain a functor from the category **gr** of finitely generated free groups to the pointed homotopy category. This allows us to view higher Hochschild homology $\mathbb{Z}^{\star r} \mapsto HH_*(B\mathbb{Z}^{\star r}; L)$ as an $\mathbb{N}$-graded object of $\mathcal{F}(\mathbf{gr}; \mathbb{Q})$, the category of functors from **gr** to $\mathbb{Q}$-vector spaces.

Our key strategy in this paper is to take into account the full structure as a functor on **gr**, rather than just the underlying family of representations of the automorphism groups $\mathrm{Aut}(\mathbb{Z}^{\star r})$, for $r \in \mathbb{N}$. Functors on the category of finitely generated free groups **gr** are relatively rigid objects; this simplifies their analysis compared to that of arbitrary representations of $\mathrm{Aut}(\mathbb{Z}^{\star r})$. Considering functors on **gr** allows us to highlight properties and study relationships in a more organised and systematic manner.

The first functorial tool used in this paper is the notion of *polynomial functors*. This notion, initially introduced by Eilenberg and MacLane [11] for functors on categories of modules, has been extended to a wider context in [18], which includes functors on the category of finitely generated free groups **gr**. Polynomiality gives a measure of the complexity of a functor, namely its polynomial degree. A functor M on **gr** being polynomial of degree d roughly means that all values of M are determined by the restrictions $M(\mathbb{Z}^{\star r})$ for $r \leq d$.

The significance of polynomiality for higher Hochschild homology is exhibited by the following, proved as part of Theorem 17.8.

Theorem 1. — *For a Γ-module L and $d \in \mathbb{N}$, the functor $HH_d(B(-); L)$ on* **gr** *has polynomial degree d.*

A polynomial functor comes equipped with a natural polynomial filtration, the form of which is completely understood for functors on **gr** (see Proposition 4.11); this plays a crucial role.

In this paper, we introduce a new functorial tool, the notion of an *outer functor* on free groups. The category of outer functors on **gr** is the full subcategory $\mathcal{F}^{\mathrm{Out}}(\mathbf{gr}; \mathbb{Q}) \subset \mathcal{F}(\mathbf{gr}; \mathbb{Q})$ of functors on which the subgroup of inner automorphisms $\mathrm{Inn}(\mathbb{Z}^{\star r}) \subseteq \mathrm{Aut}(\mathbb{Z}^{\star r})$ acts trivially, for each $r \in \mathbb{N}$. Higher Hochschild homology (for certain coefficients) gives a source of highly non-trivial outer functors; for example, as a particular case of Proposition 13.20, we obtain the following.

Proposition 2. — *For a commutative $\mathbb{Q}$-algebra A and $d \in \mathbb{N}$, $HH_d(B(-); \mathcal{L}(A, A))$ is an outer functor on* **gr**.

Indeed, whenever the coefficients L arise from a functor defined on *unpointed* finite sets (for example, $L = \mathcal{L}(A, A)$), higher Hochschild homology $X \mapsto HH_*(X; L)$ is a functor on the unpointed homotopy category. In this situation, it follows that the action of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{\star r})$ on $HH_*(B\mathbb{Z}^{\star r}; L)$ factors across the outer automorphism group $\mathrm{Out}(\mathbb{Z}^{\star r})$, as observed by Turchin and Willwacher [41]. Proposition 13.20 is the functorial analogue of this observation.

As above, higher Hochschild homology gives examples of objects in $\mathcal{F}^{\mathrm{Out}}(\mathbf{gr}; \mathbb{Q})$, but this is not the only context in which such functors arise naturally (see the end of the Introduction) and an in-depth study of them is of independent interest.

The category $\mathcal{F}^{\mathrm{Out}}(\mathbf{gr};\mathbb{Q})$ of outer functors is crucial in our work, as is the right adjoint

$$\omega : \mathcal{F}(\mathbf{gr};\mathbb{Q}) \to \mathcal{F}^{\mathrm{Out}}(\mathbf{gr};\mathbb{Q})$$

to the inclusion $\mathcal{F}^{\mathrm{Out}}(\mathbf{gr};\mathbb{Q}) \subset \mathcal{F}(\mathbf{gr};\mathbb{Q})$. This arises naturally in the study of higher Hochschild homology, as is explained below.

The third functorial tool used in this paper is provided by *exponential functors*; i.e. symmetric monoidal functors on **gr**. We recall, in Theorem 3.5, the equivalence of categories between the category of exponential functors on **gr**, $\mathcal{F}^{\exp}(\mathbf{gr};\mathbb{Q})$, and the category of commutative Hopf algebras on $\mathbb{Q}$, $\mathbf{Hopf}^{\mathrm{com}}(\mathbb{Q}\text{-mod})$, induced by the functor $\Psi : \mathbf{Hopf}^{\mathrm{com}}(\mathbb{Q}\text{-mod}) \to \mathcal{F}^{\exp}(\mathbf{gr};\mathbb{Q})$ given on objects by $\Psi H(\mathbb{Z}^{\star n}) = H^{\otimes n}$. (The result is more general: the category $\mathbb{Q}$-mod of $\mathbb{Q}$-vector spaces can be replaced by any suitable symmetric monoidal category.)

To state our results on higher Hochschild homology, we must first introduce the appropriate coefficients. As in the work of Turchin and Willwacher [40], we consider the square-zero extension $A_V := \mathbb{Q} \oplus V$ of $\mathbb{Q}$ by a finite-dimensional $\mathbb{Q}$-vector space V. For $V = \mathbb{Q}$, this corresponds to the dual numbers $\mathbb{Q}[\varepsilon]$.

The main problem that we address is the following.

Problem 3. — *Identify the following functors on* **gr***:*

$$HH_*(B(-);\mathcal{L}(A_V,\mathbb{Q}))$$
$$HH_*(B(-);\mathcal{L}(A_V,A_V)).$$

A key observation is that the association $V \mapsto A_V$ is a functor of $V \in \mathsf{Ob}\,\mathbf{mod}_{\mathbb{Q}}$, where $\mathbf{mod}_{\mathbb{Q}}$ is the category of finite-dimensional $\mathbb{Q}$-vector spaces. This naturality is a crucial ingredient in studying these structures; it also allows us to bring into play the Schur correspondence between representations of the symmetric groups and functors on $\mathbf{mod}_{\mathbb{Q}}$. This leads us to study the functors $HH_*(B(-);\mathcal{L}(A_V,\mathbb{Q}))$ and $HH_*(B(-);\mathcal{L}(A_V,A_V))$ on **gr** naturally with respect to V.

The functor $HH_*(B(-);\mathcal{L}(A_V,\mathbb{Q}))$ is exponential (see Proposition 13.18), and we identify the corresponding commutative Hopf algebra as follows. Denoting by sV the homological suspension of $V \in \mathsf{Ob}\,\mathbf{mod}_{\mathbb{Q}}$, we consider the tensor coalgebra $T_{\mathrm{coalg}}(sV)$ as a graded-commutative Hopf algebra with respect to the shuffle product. This then yields the exponential functor $\Psi(T_{\mathrm{coalg}}(sV))$. In Theorem 15.1, we obtain the following description.

Theorem 4. — *For $V \in \mathsf{Ob}\,\mathbf{mod}_{\mathbb{Q}}$, there is a natural isomorphism of functors with values in graded-commutative $\mathbb{Q}$-algebras:*

$$HH_*(B(-);\mathcal{L}(A_V,\mathbb{Q})) \cong \Psi(T_{\mathrm{coalg}}(sV)),$$

where $HH_*(B(-);\mathcal{L}(A_V,\mathbb{Q}))$ is equipped with the shuffle product. This is natural with respect to V.

The functor $HH_*(B(-);\mathcal{L}(A_V,A_V))$ is then studied by using the long exact sequence associated to the short exact sequence of Γ-module coefficients:

$$0 \to \mathcal{L}(A_V,\mathbb{Q}) \otimes V \to \mathcal{L}(A_V,A_V) \to \mathcal{L}(A_V,\mathbb{Q}) \to 0,$$

in which the surjection $\mathcal{L}(A_V,A_V) \to \mathcal{L}(A_V,\mathbb{Q})$ is induced by the augmentation $A_V \twoheadrightarrow \mathbb{Q}$. The connecting morphism of this long exact sequence is identified in Proposition 16.4 as being the coadjoint coaction

$$\overline{\text{coad}} : \Psi(T_{\text{coalg}}(sV)) \to sV \otimes \Psi(T_{\text{coalg}}(sV)),$$

which fits into the following exact sequence:

(1.1)

$$0 \to \omega\Psi T_{\text{coalg}}(sV) \to \Psi T_{\text{coalg}}(sV) \xrightarrow{\overline{\text{coad}}} \Psi T_{\text{coalg}}(sV) \otimes sV \to \text{coker } (\overline{\text{coad}}) \to 0.$$

This leads to Theorem 16.5, in which the $\mathbb{N}$ accounts for the homological grading.

Theorem 5. — *For $V \in \mathsf{Ob}\, \mathbf{mod}_{\mathbb{Q}}$, there is a natural isomorphism in $\mathcal{F}(\mathbb{N} \times \mathbf{gr};\mathbb{Q})$:*

$$HH_*\big(B(-);\mathcal{L}(A_V,A_V)\big) \cong \omega\Psi(T_{\text{coalg}}(sV)) \oplus \text{coker } (\overline{\text{coad}}_{T_{\text{coalg}}(sV)})[-1],$$

where $[-1]$ denotes the shift in homological degree. These identifications are natural with respect to V.

It follows that, to understand these higher Hochschild homology functors, it suffices to understand the functor $V \mapsto \Psi T_{\text{coalg}}(sV)$ and the outer functor $V \mapsto \omega\Psi T_{\text{coalg}}(sV)$, since the remaining term can be derived from these using the exact sequence (1.1).

For $V = \mathbb{Q}$, Theorem 5 extends and gives a functorial interpretation of results of [41] for the dual numbers (cf. [41, Theorem 1] and the direct sum decomposition given in [41, Section 2.2]). See Remark 16.12 for more details.

1.1. Combinatorial coefficients. — To prove Theorem 5, we use the functoriality with respect to V evoked above. The Schur correspondence gives

$$\text{Inj}^{\Gamma} \leftrightarrow \mathcal{L}(A_V;\mathbb{Q})$$
$$\vartheta^*\text{Inj}^{\mathbf{Fin}} \leftrightarrow \mathcal{L}(A_V,A_V),$$

where Inj^{Γ} and $\vartheta^*\text{Inj}^{\mathbf{Fin}}$ are Γ-modules introduced in Section 14, taking values in $\mathbf{\Sigma}$-modules, where $\mathbf{\Sigma}$ is the category of finite sets and bijections (which serves to encode the family of symmetric groups $\{\mathfrak{S}_d \,|\, d \in \mathbb{N}\}$).