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Yota Shamoto

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

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STOKES STRUCTURE OF MILD DIFFERENCE MODULES

by Yota Shamoto

Abstract. — We introduce a category of filtered sheaves on a circle to describe the Stokes phenomenon of linear difference equations with mild singularity. The main result is a mild difference analog of the Riemann–Hilbert correspondence for germs of meromorphic connections in one complex variable by Deligne–Malgrange.

Résumé (*Structure de Stokes des modules à différences modérées*). — Nous introduisons une catégorie de faisceaux filtrés sur un cercle pour décrire le phénomène de Stokes des équations aux différences linéaires avec une légère singularité. Le résultat principal est un analogue de légère différence de la correspondance de Riemann-Hilbert pour les germes de connexions méromorphes dans une variable complexe par Deligne-Malgrange.

1. Introduction

1.1. Mild difference modules. — Let $\mathscr{O}_t := \mathbb{C}\{t\}$ be the ring of convergent power series in a variable t. Let $\mathscr{O}_t(*0) = \mathbb{C}(\{t\})$ be the quotient field. Let ϕ_t be the automorphism on $\mathscr{O}_t(*0)$ defined as $\phi_t(f)(t) := f(\frac{t}{1+t})$. If we set $s = t^{-1}$, we have $\phi_t(f)(s) = f(s+1)$. By *a difference module* (over the difference

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Yota Shamoto, Faculty of Science and Engineering, Yamato University, Katayama-cho 2-5-1, Suita City, Osaka, Japan • *E-mail :* shamoto.yota@yamato-u.ac.jp

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field $(\mathscr{O}_t(*0), \phi_t)$), we mean a pair $(\mathscr{M}, \psi)$ consisting of a finite-dimensional $\mathscr{O}_t(*0)$-vector space $\mathscr{M}$ and an automorphism $\psi\colon \mathscr{M} \to \mathscr{M}$ of $\mathbb{C}$-vector spaces satisfying the relation $\psi(fv) = \phi_t(f)\psi(v)$ for any $f \in \mathscr{O}_t(*0)$ and $v \in \mathscr{M}$.

There is a class of difference modules called *mild* [10, §9]. A difference module is called *mild* if it is isomorphic to a module of the form $(\mathscr{O}_t(*0)^{\oplus r}, A(t)\phi_t^{\oplus r})$, where $A(t)$ has entries in $\mathscr{O}_t$, and the constant term $A(0)$ is invertible.

The purpose of this paper is to establish the Riemann–Hilbert correspondence for mild difference modules as an analog of that for germs of meromorphic connections in one complex variable by Deligne–Malgrange [3, 9]. See also [11].

1.2. Stokes filtered locally free sheaves for mild difference modules. — To formulate the Riemann–Hilbert correspondence for mild difference modules, we introduce the notion of Stokes filtered locally free sheaves for difference modules in §3. We explain the notion briefly, comparing it with the case of meromorphic connections.

In the case of germs of meromorphic connections, we consider the notion of a Stokes filtered local system on $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. It is a pair $(\mathcal{L}, \mathcal{L}_\bullet)$ of a local system $\mathcal{L}$ of finite-dimensional $\mathbb{C}$-vector spaces on S^1 and a filtration $\mathcal{L}_\bullet$ on $\mathcal{L}$ indexed by a sheaf $\mathcal{I} = \bigcup_{m\geq 0} z^{-\frac{1}{m}}\mathbb{C}[z^{-\frac{1}{m}}]$ of ordered abelian groups.

The Deligne–Malgrange theorem claims that there is an equivalence (called the Riemann–Hilbert functor) between the category of germs of meromorphic connections and the category of Stokes filtered local systems on S^1.

When a Stokes filtered local system $(\mathcal{L}, \mathcal{L}_\bullet)$ corresponds to a germ of a meromorphic connection $\mathcal{M}$ by the Riemann–Hilbert functor, the sheaf $\mathcal{L}$ is regarded as the sheaf of flat sections of $\mathcal{M}$ on sectors- and the filtration $\mathcal{L}_\bullet$ describes the growth rate of the sections. The filtration $\mathcal{L}_\bullet$ is called the Stokes filtration on $\mathcal{L}$ since the relation between the splittings of $\mathcal{L}_\bullet$ on different domains describes the classical Stokes phenomenon of the solutions of the differential equation associated to $\mathcal{M}$.

In the case of difference modules, we consider a locally free sheaf $\mathscr{L}$ over a sheaf $\mathscr{A}_{\mathrm{per}}$ of rings over S^1. Here, $\mathscr{A}_{\mathrm{per}}$ is the sheaf of rings over S^1 defined as follows:

$$\mathscr{A}_{\mathrm{per}}(U) = \begin{cases} \mathbb{C}(\{u^{-1}\}) & (U \subset (0,\pi), U \neq \emptyset) \\ \mathbb{C}(\{u\}) & (U \subset (-\pi, 0), U \neq \emptyset) \\ \mathbb{C}[u^{\pm 1}] & (U \cap \{e^{i\pi}, e^0\} \neq \emptyset), \end{cases}$$

where U is assumed to be connected, and we set $(a,b) := \{e^{\mathtt{i}\theta} \in S^1 \mid a < \theta < b\}$ for $a, b \in \mathbb{R}$ with $a < b$. If we put $u = \exp(2\pi\mathtt{i}t^{-1})$, we can regard $\mathscr{A}_{\mathrm{per}}$ as a sheaf of rings of a certain class of ϕ_t-invariant (or periodic with respect to $s \mapsto s+1$) functions (see §2 for more details).

Then, we define a filtration $\mathscr{L}_\bullet$ on $\mathscr{L}$ indexed by a sheaf of ordered abelian groups. We call it a Stokes filtration. It will turn out that this filtration describes the growth rate of the solutions of the difference equation associated to the difference module. A new feature of the filtration is the compatibility of the action of u with the filtration:

$$u\mathscr{L}_{\leqslant\mathfrak{a}} = \mathscr{L}_{\leqslant\mathfrak{a}+2\pi\mathtt{i}t^{-1}},$$

where $\mathfrak{a}$ is an arbitrary index (see §3 for more details).

It is worth mentioning that a Stokes filtered $\mathscr{A}_{\mathrm{per}}$-module can be non-graded even if it is rank one as a free $\mathscr{A}_{\mathrm{per}}$-module. This point will be explained in §4.7.

1.3. Main result. — Let $\mathsf{Diffc}^{\mathrm{mild}}$ be the category of mild difference modules. Let $\mathsf{St}(\mathscr{A}_{\mathrm{per}})$ be the category of Stokes filtered locally free $\mathscr{A}_{\mathrm{per}}$-modules. Then, we can state the main result of the present paper.

THEOREM 1.1 (Theorem 4.17). — *There is a functor* $\mathrm{RH}\colon \mathsf{Diffc}^{\mathrm{mild}} \to \mathsf{St}(\mathscr{A}_{\mathrm{per}})$, *which is an equivalence of categories.*

This result is analogous to that of Deligne–Malgrange [3, 9] (See [11, Theorem 5.8]). The proof of this theorem is similar to the one that can be found in [11]. The main difference is the definition of the functor RH. See Remark 4.11 for details.

The author hopes that this result contributes to the intrinsic understanding of linear difference modules. In particular, it would be interesting to use the result to describe the Stokes structure of the Mellin transformation of a holonomic $\mathcal{D}$-module concerning recent progress [1, 6, 5, 4] in the study of the Mellin transformations (see Remark 4.4).

1.4. Outline of the paper. — In §2, we prepare some notions used throughout the paper. In §3, we introduce the notion of Stokes filtered $\mathscr{A}_{\mathrm{per}}$-modules. In §4, we formulate and prove the main theorem assuming a theorem proved in §5.

2. Preliminaries

In this section, we prepare some notions used throughout the paper.

2.1. An automorphism on a projective line. — Let $\mathbb{C}$ be the set of complex numbers. Set $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Natural inclusion is denoted by $\jmath\colon \mathbb{C}^* \to \mathbb{C}$. When we distinguish a variable such as t, we use the symbols $\mathbb{C}_t$ and $\mathbb{C}_t^*$. Let $S^1 = \{e^{\mathtt{i}\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$ be the unit circle, where we set $\mathtt{i} = \sqrt{-1}$. For two real numbers a, b with $a < b$, we set $(a, b) := \{e^{\mathtt{i}\theta} \in S^1 \mid a < \theta < b\}$.

2.1.1. *Real blowing up.* — We set

$$\widetilde{\mathbb{C}} = \{(t, e^{\mathtt{i}\theta}) \in \mathbb{C} \times S^1 \mid t = |t|e^{\mathtt{i}\theta}\},$$

which is called *the real blowing up of* $\mathbb{C}$ *at the origin.* When we distinguish a variable such as t, we use the notation $\widetilde{\mathbb{C}}_t$. There are maps $\varpi\colon \widetilde{\mathbb{C}} \to \mathbb{C}$, $\widetilde{\jmath}\colon \mathbb{C}^* \hookrightarrow \widetilde{\mathbb{C}}$, and $\widetilde{\imath}\colon S^1 \hookrightarrow \widetilde{\mathbb{C}}$ defined by $\varpi(t, e^{\mathtt{i}\theta}) = t$, $\widetilde{\jmath}(t) = (t, t/|t|)$, and $\widetilde{\imath}(e^{\mathtt{i}\theta}) = (0, e^{\mathtt{i}\theta})$, respectively. We sometimes denote the boundary of $\widetilde{\mathbb{C}}_t$ by S^1_t to distinguish a variable such as t.

2.1.2. *Unit disc.* — Let $\Delta \coloneqq \{t \in \mathbb{C} \mid |t| < 1\}$ be a unit open disc. We set $\Delta^* \coloneqq \Delta \setminus \{0\}$ and $\widetilde{\Delta} \coloneqq \{(t, e^{\mathtt{i}\theta}) \in \widetilde{\mathbb{C}} \mid t \in \Delta\}$. Let $\varpi_\Delta\colon \widetilde{\Delta} \to \Delta$ be the projection. When we distinguish a variable such as t, we use the notations Δ_t, Δ^*_t and $\widetilde{\Delta}_t$. The natural inclusions are denoted by $\jmath_\Delta\colon \Delta^* \to \Delta$, $\widetilde{\imath}_\Delta\colon S^1 \to \widetilde{\Delta}$, and $\widetilde{\jmath}_\Delta\colon \Delta^* \to \widetilde{\Delta}$. Let $\varphi_t\colon \Delta_t \to \mathbb{C}_t$ be the holomorphic function defined as

$$\varphi_t(t) \coloneqq \frac{t}{1+t}.$$

This map uniquely extends to a continuous map $\widetilde{\varphi}_t\colon \widetilde{\Delta}_t \to \widetilde{\mathbb{C}}_t$.

2.1.3. *Another coordinate.* — Let $\mathbb{C}_s$ be a complex plane with a coordinate s. When we use the two complex variables s and t, we implicitly assume the relation

$$s = t^{-1}. \tag{1}$$

In other words, we consider the complex projective line $\mathbb{P}^1$ covered by two open subsets $\mathbb{C}_s$ and $\mathbb{C}_t$ with the relation (1). Let $\varphi_s\colon \mathbb{C}_s \to \mathbb{C}_s$ be the map defined as

$$\varphi_s(s) = s + 1.$$

Under the relation (1), the map φ_s coincides with the map φ_t on the domain $\Delta^*_t = \{s \in \mathbb{C}_s \mid |s| > 1\}$. Hence the maps φ_s and φ_t are glued to an automorphism on $\mathbb{P}^1$.

2.2. Sheaves of periodic functions. — For a point x in a topological space X and a sheaf $\mathscr{F}$ on X, let $\mathscr{F}_x$ denote the set of germs of $\mathscr{F}$ at x. For a continuous map $f\colon X \to Y$ between topological spaces, f_* denotes the pushing forward of sheaves, and f^{-1} denotes the pulling back of sheaves. If X is a complex manifold, let $\mathscr{O}_X$ denote the sheaf of holomorphic functions on X. If X is a Riemann surface and D is a finite set of points, then let $\mathscr{O}_X(*D)$ denote the sheaf of meromorphic functions on X whose poles are contained in D.

2.2.1. *Functions with fixed asymptotic behavior.* — Using the notations in §2.1.1, set

$$\widetilde{\mathscr{O}} := \widetilde{\imath}^{-1}\widetilde{\jmath}_*\mathscr{O}_{\mathbb{C}^*},$$

which is a sheaf on S^1. There are subsheaves $\mathscr{A}^{\leqslant 0}$, and $\mathscr{A}^{<0}$ of $\widetilde{\mathscr{O}}$ characterized by their asymptotic behavior as follows (see [11] for precise definitions).

- $\mathscr{A}^{\leqslant 0}$ is the sheaf of holomorphic functions *which are of moderate growth.*
- $\mathscr{A}^{<0}$ is the sheaf of holomorphic functions *which are of rapid decay.*

Here, a local section $f \in \widetilde{\mathscr{O}}$ on $U \subset S^1$, represented by a holomorphic function $\widetilde{f} \in \widetilde{\jmath}_*\mathscr{O}_{\mathbb{C}^*}$ on an open subset $\widetilde{U} \subset \widetilde{\mathbb{C}}_t$ with $U = S^1 \cap \widetilde{U}$ is *of moderate growth* if for any compact subset $K \subset \widetilde{U}$, there exist constants $C_K > 0$ and $N_K \geq 0$ such that

$$|\widetilde{f}(z)| \leq C_K |t(z)|^{-N_K} \text{ for any } z \in K \setminus U.$$

Similarly, f is *of rapid decay* if for any compact $K \subset \widetilde{U}$ and any $N \geq 0$, there exists a constant $C_{K,N} > 0$ such that

$$|\widetilde{f}(z)| \leq C_{K,N} |t(z)|^N \text{ for any } z \in K \setminus U.$$

To emphasize a coordinate function such as t, we use the notation $\widetilde{\mathscr{O}}_t$, $\mathscr{A}_t^{\leqslant 0}$, etc.

2.2.2. *Periodic functions.* — The map $\widetilde{\varphi}_t \colon \widetilde{\Delta}_t \to \widetilde{\mathbb{C}}_t$ defined in §2.1.2 naturally induces a morphism $\widetilde{\varphi}_t^* \colon \widetilde{\jmath}_*\mathscr{O}_{\mathbb{C}^*} \to \widetilde{\varphi}_{t*}\widetilde{\jmath}_{\Delta*}\mathscr{O}_{\Delta^*}$ defined as the composition with the map φ restricted to a suitable open subset. We then set

$$\widetilde{\phi}_t := \widetilde{\imath}^{-1}(\widetilde{\varphi}_t^*) \colon \widetilde{\mathscr{O}}_t \to \widetilde{\mathscr{O}}_t,$$

where we used the relation $\widetilde{\varphi}_t \circ \widetilde{\imath}_\Delta = \widetilde{\imath}$. Note that the subsheaves $\mathscr{A}_t^{\leqslant 0}$ and $\mathscr{A}_t^{<0}$ are invariant under the automorphism $\widetilde{\phi}_t$. We then set $\nabla_{\widetilde{\phi}_t} := \widetilde{\phi}_t - \mathrm{id}_{\widetilde{\mathscr{O}}}$. The restrictions of $\nabla_{\widetilde{\phi}_t}$ to $\mathscr{A}_t^{\leqslant 0}$ and $\mathscr{A}_t^{<0}$ are also denoted by the same symbol. Set

$$u := \exp(2\pi\mathtt{i}s) = \exp(2\pi\mathtt{i}t^{-1}) \tag{2}$$

and $v = u^{-1}$. Then, we have the equality $\nabla_{\widetilde{\phi}_t}(u) = 0$.

Definition 2.1 (Sheaves of periodic functions). — Let $\widetilde{\mathscr{O}}_{\mathrm{per}}$, $\mathscr{A}_{\mathrm{per}}^{<0}$, and $\mathscr{A}_{\mathrm{per}}^{\leqslant 0}$ denote the kernel of the operator $\nabla_{\widetilde{\phi}_t}$ on $\widetilde{\mathscr{O}}$, $\mathscr{A}^{<0}$ and $\mathscr{A}^{\leqslant 0}$, respectively.

Lemma 2.2 (cf. [10, p. 117–118]). — *For a non-empty connected open subset $U \subset S_t^1$, we have the following descriptions of $\widetilde{\mathscr{O}}_{\mathrm{per}}(U)$, $\mathscr{A}_{\mathrm{per}}^{\leqslant 0}(U)$, and $\mathscr{A}_{\mathrm{per}}^{<0}(U)$:*

- *If $U \subset (0, \pi)$, then*

$$\widetilde{\mathscr{O}}_{\mathrm{per}}(U) = (\jmath_*\mathscr{O}_{\mathbb{C}_v^*})_0, \quad \mathscr{A}_{\mathrm{per}}^{\leqslant 0}(U) = \mathscr{O}_v, \quad \mathscr{A}_{\mathrm{per}}^{<0}(U) = v\mathscr{O}_v.$$