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SUBGROUP DYNAMICS AND C^*-SIMPLICITY OF GROUPS OF HOMEOMORPHISMS

BY Adrien LE BOUDEC AND Nicolás MATTE BON

Abstract. – We study the uniformly recurrent subgroups of groups acting by homeomorphisms on a topological space. We prove a general result relating uniformly recurrent subgroups to rigid stabilizers of the action, and deduce a C^*-simplicity criterion based on the non-amenability of rigid stabilizers. As an application, we show that Thompson's group V is C^*-simple, as well as groups of piecewise projective homeomorphisms of the real line. This provides examples of finitely presented C^*-simple groups without free subgroups. We prove that a branch group is either amenable or C^*-simple. We also prove the converse of a result of Haagerup and Olesen: if Thompson's group F is non-amenable, then Thompson's group T must be C^*-simple. Our results further provide sufficient conditions on a group of homeomorphisms under which uniformly recurrent subgroups can be completely classified. This applies to Thompson's groups F, T and V, for which we also deduce rigidity results for their minimal actions on compact spaces.

Résumé. – Nous étudions les sous-groupes uniformément récurrents de groupes agissant par homéomorphismes sur un espace topologique. Nous prouvons un résultat général reliant les sous-groupes uniformément récurrents aux stabilisateurs rigides de l'action, et en déduisons un critère de C^*-simplicité basé sur la non moyennabilité des stabilisateurs rigides. Comme application, nous prouvons que le groupe de Thompson V est C^*-simple, de même que certains groupes d'homéomorphismes projectifs par morceaux de la droite réelle. Cela fournit des exemples de groupes finiment présentés qui sont C^*-simples et sans sous-groupes libres. Nous prouvons qu'un groupe branché est soit moyennable, soit C^*-simple. Nous prouvons également la réciproque d'un résultat de Haagerup et Olesen: si le groupe de Thompson F n'est pas moyennable alors le groupe de Thompson T est C^*-simple. Nos résultats fournissent de plus des conditions suffisantes sur un groupe d'homéomorphismes sous lesquelles les sous-groupes uniformément récurrents sont complètement compris. Cela s'applique aux groupes de Thompson, pour lesquels nous déduisons également des résultats de rigidité sur leurs actions sur des espaces compacts.

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1. Introduction

Let G be a second countable locally compact group. The set $\mathrm{Sub}(G)$ of all closed subgroups of G admits a natural topology, defined by Chabauty in [17]. This topology turns $\mathrm{Sub}(G)$ into a compact metrizable space, on which G acts continuously by conjugation.

The study of G-invariant Borel probability measures on $\mathrm{Sub}(G)$, named *invariant random subgroups* (IRS's) after [2], is a fast-developing topic [2, 1, 27, 3]. In this paper we are interested in their topological counterparts, called *uniformly recurrent subgroups* (URSs) [24]. A uniformly recurrent subgroup is a closed, minimal, G-invariant subset $\mathcal{H} \subset \mathrm{Sub}(G)$. We will denote by $\mathrm{URS}(G)$ the set of uniformly recurrent subgroups of G. Every normal subgroup $N \trianglelefteq G$ gives rise to a URS of G, namely the singleton $\{N\}$. More interesting examples arise from minimal actions on compact spaces: if G acts minimally on a compact space X, then the closure of all point stabilizers in $\mathrm{Sub}(G)$ contains a unique URS, called the *stabilizer URS* of $G \curvearrowright X$ [24].

When G has only countably many subgroups (e.g., if G is polycyclic), every IRS of G is atomic and every URS of G is finite, as follows from a standard Baire argument. Leaving aside this specific situation, there are important families of groups for which a precise description of the space $\mathrm{IRS}(G)$ has been obtained. Considerably less is known about URSs. For example if G is a lattice in a higher rank simple Lie group, the normal subgroup structure of G is described by Margulis' Normal Subgroups Theorem. While Stuck-Zimmer's Theorem [64] generalizes Margulis' NST to IRS's, it is an open question whether a similar result holds for URSs of higher rank lattices, even for the particular case of $\mathrm{SL}(3, \mathbb{Z})$ [24, Problem 5.4].

1.1. Micro-supported actions

In this paper we study the space $\mathrm{URS}(G)$ for countable groups G admitting a faithful action $G \curvearrowright X$ on a topological space X such that for every non-empty open set $U \subset X$, the *rigid stabilizer* G_U, i.e., the pointwise stabilizer of $X \setminus U$ in G, is non-trivial. Following [16], such an action will be called *micro-supported*. Note that this implies in particular that X has no isolated points.

The class of groups admitting a micro-supported action includes Thompson's groups $F < T < V$ and many of their generalizations, groups of piecewise projective homeomorphisms of the real line [49], piecewise prescribed tree automorphism groups [41], branch groups (viewed as groups of homeomorphisms of the boundary of the rooted tree) [4], and topological full groups acting minimally on the Cantor set. These groups have uncountably many subgroups, and many examples in this class have a rich subgroup structure.

Our first result shows that many algebraic or analytic properties of rigid stabilizers are inherited by the uniformly recurrent subgroups of G. In the following theorem and everywhere in the paper, a uniformly recurrent subgroup $\mathcal{H} \in \mathrm{URS}(G)$ is said to have a group property if every $H \in \mathcal{H}$ has the corresponding property.

Theorem 1.1 (see also Theorem 3.5). – *Let G be a countable group of homeomorphisms of a Hausdorff space X. Assume that for every non-empty open set $U \subset X$, the rigid stabilizer G_U is non-amenable (respectively contains free subgroups, is not elementary amenable, is* *not virtually solvable, is not locally finite). Then every non-trivial uniformly recurrent subgroup of G has the same property.*

This result has applications to the study of C^*-simplicity; see Section 1.2.

We obtain stronger conclusions on the uniformly recurrent subgroups of G under additional assumptions on the action of G on X. Recall that when X is compact and $G \curvearrowright X$ is minimal, the closure of all point stabilizers in $\mathrm{Sub}(G)$ contains a unique URS, called the *stabilizer URS* of $G \curvearrowright X$, and denoted $\mathcal{S}_G(X)$ [24] (see Section 2 for details). The following result provides sufficient conditions under which $\mathcal{S}_G(X)$ turns out to be the unique URS of G, apart from the points $\{1\}$ and $\{G\}$ (hereafter denoted 1 and G). We say that $G \curvearrowright X$ is an *extreme boundary action* if X is compact and the action is minimal and extremely proximal (see §2.1 for the definition of an extremely proximal action).

THEOREM 1.2. – *Let X be a compact Hausdorff space, and let G be a countable group of homeomorphisms of X. Assume that the following conditions are satisfied:*

(i) *$G \curvearrowright X$ is an extreme boundary action;*
(ii) *there is a basis for the topology consisting of open sets $U \subset X$ such that the rigid stabilizer G_U admits no non-trivial finite or abelian quotients;*
(iii) *the point stabilizers for the action $G \curvearrowright X$ are maximal subgroups of G.*

Then the only uniformly recurrent subgroups of G are 1, *G and* $\mathcal{S}_G(X)$.

We actually prove a more general result, see Corollary 3.12. Examples of groups to which this result applies are Thompson's groups T and V, as well as examples in the family of groups acting on trees $G(F, F')$ (see Section 1.3 and Section 1.5). In particular, this provides examples of finitely generated groups G (with uncountably many subgroups) for which the space $\mathrm{URS}(G)$ is completely understood. In the case of Thompson's groups, we deduce from this lack of URSs rigidity results about their minimal actions on compact spaces (see §1.3).

1.2. Application to C^*-simplicity

A group G is said to be C^*-*simple* if its reduced C^*-algebra $C^*_{\mathrm{red}}(G)$ is simple. This property naturally arises in the study of unitary representations: G is C^*-simple if and only if every unitary representation of G that is weakly contained in the left-regular representation λ_G is actually weakly equivalent to λ_G [33]. Since amenability of a group G is characterized by the fact that the trivial representation of G is weakly contained in λ_G, a non-trivial amenable group is never C^*-simple.

The first historical C^*-simplicity result was Powers' proof that the reduced C^*-algebra of the free group $\mathbb{F}_2$ is simple [60]. The methods employed by Powers have then been generalized in several different ways, and various classes of groups have been shown to be C^*-simple. We refer to [33, Proposition 11] (see also the references given there), and to Corollary 12 therein for a list of important examples of groups to which these methods have been applied.

Problems related to C^*-simplicity recently received new attention [38, 8, 61, 41, 39, 31]. A characterization of C^*-simplicity in terms of boundary actions was obtained by Kalantar and Kennedy: a countable group G is C^*-simple if and only if G acts topologically freely on its Furstenberg boundary; equivalently, G admits some topologically free boundary action [38] (we recall the terminology in §2.1). By developing a systematic approach based on