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Karel Pravda-Starov

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PSEUDO-SPECTRUM FOR A CLASS OF SEMI-CLASSICAL OPERATORS

by Karel Pravda-Starov

Abstract. — We study in this paper a notion of pseudo-spectrum in the semi-classical setting called injectivity pseudo-spectrum. The injectivity pseudo-spectrum is a subset of points in the complex plane where there exist some quasi-modes with a precise rate of decay. For that reason, these values can be considered as some 'almost eigenvalues' in the semi-classical limit. We are interested here in studying the absence of injectivity pseudo-spectrum, which is characterized by a global a priori estimate. We prove in this paper a sharp global subelliptic a priori estimate for a class of pseudo-differential operators with respect to the regularity of their symbols. Our main result extends the a priori estimate of Dencker, Sjöstrand and Zworski for a class of pseudo-differential operators with symbols of limited smoothness violating the condition (P).

Résumé (*Pseudo-spectre d'une classe d'opérateurs semi-classiques*)

Nous étudions dans cet article une notion de pseudo-spectre semi-classique appelée pseudo-spectre d'injectivité. Le pseudo-spectre d'injectivité d'un opérateur désigne l'ensemble des points du plan complexe qui sont des « presque valeurs propres » dans l'asymptotique semi-classique, au sens où il existe en ces points des quasi-modes semi-classiques avec des taux précis de décroissance. Nous nous intéressons ici à l'étude de l'absence de pseudo-spectre d'injectivité, et nous démontrons une estimation sous-elliptique globale pour une classe d'opérateurs pseudo-différentiels dont les symboles ont une régularité limitée. Ce résultat généralise dans un cadre de régularité limitée l'estimation a priori démontrée par Dencker, Sjöstrand et Zworski pour une classe d'opérateurs pseudo-différentiels violant la condition (P).

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Karel Pravda-Starov, Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA • *E-mail :* karel@math.ucla.edu

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1. Introduction

1.1. Miscellaneous facts about pseudo-spectrum. — In recent years, there has been a lot of interest in studying the pseudo-spectrum of non-self-adjoint operators. We first recall some classical and known facts about pseudo-spectrum. The study of pseudo-spectrum has been initiated by noticing that for certain problems of science and engineering involving non-self-adjoint operators, the predictions suggested by spectral analysis do not match with the numerical simulations. To supplement the lack of information given by the spectrum, some new subsets of the complex plane called pseudo-spectra have been defined. The main idea is to study not only points where the resolvent is not defined, i.e., the spectrum, but also where the resolvent is large in norm.

We refer the reader to Trefethen's article [11] (1) for the definition of the ε-pseudo-spectrum $\Lambda_\varepsilon(A)$ of a matrix or an operator A,

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C}, \|(zI - A)^{-1}\| \geq \varepsilon^{-1}\}.$$

By convention, we write $\|(zI - A)^{-1}\| = +\infty$ if z belongs to the spectrum of A. The ε-pseudo-spectrum of A is non-decreasing with ε. All these subsets contain the spectrum of the operator. In an equivalent way, pseudo-spectra can be defined in term of the spectra of perturbations. Indeed, for any matrix we have

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C},\ z \in \sigma(A + B) \text{ for some } B \text{ with } \|B\| \leq \varepsilon\}.$$

It follows that a number z belongs to the ε-pseudo-spectrum of A if and only if it belongs to the spectrum of some perturbed operator $A + B$ with $\|B\| \leq \varepsilon$. From this second description, we understand the interest in studying such subsets. Indeed, if we want to compute numerically some eigenvalues, we start by discretizing the operator. This discretization and inevitable round-off errors will generate some perturbations of the initial operator. Eventually, algorithms for eigenvalues computing determine the eigenvalues of a perturbation of the initial operator, i.e., a value in some ε-pseudo-spectrum and not necessarily a spectral value. In the self-adjoint case, the spectrum is stable under small perturbations. In fact, this stability is a consequence of the spectral theorem. The spectral theorem implies that $\Lambda_\varepsilon(A)$ is exactly the set of points in $\mathbb{C}$ at distance less than or equal to ε from the spectrum of A. However this property of stability is not true in the non-self-adjoint case in which the spectrum could be very unstable under small perturbations. To illustrate this fact, we recall a suggestive example pointed out by Davies in [3] and Zworski in [12].

(1) The reader will find in this paper more details about interest, history and general properties of pseudo-spectra.

EXAMPLE. — *Let us consider the rotated harmonic oscillator in one dimension*

$$P_\alpha = -\frac{d^2}{dx^2} + e^{i\alpha}x^2 \text{ where } -\pi < \alpha < \pi.$$

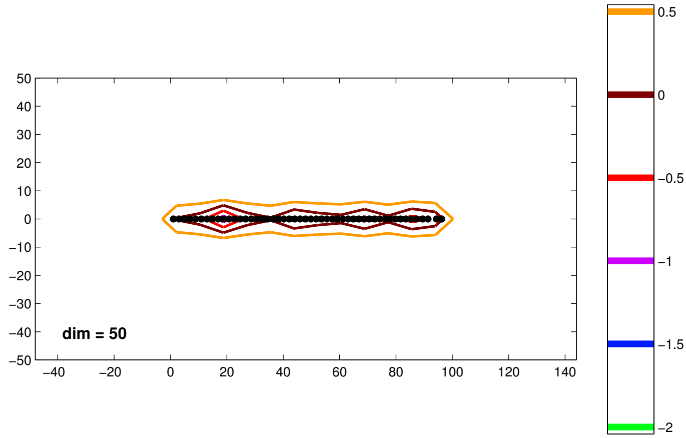

FIGURE 1. Computation of ε-pseudo-spectra for the rotated harmonic oscillator P_α with $\alpha = 0$. The right column gives the corresponding values of $\log_{10}\varepsilon$.

This operator P_α is self-adjoint only for $\alpha = 0$. Its spectrum is composed of the following eigenvalues (see Theorem 3.3 in [6]),

$$e^{i\frac{\alpha}{2}}(2n+1),\ n \in \mathbb{N}.$$

We can try to compute numerically the spectrum and some ε-pseudo-spectra for some small values of the parameter ε. Computations are performed on the discretization

$$\big((P_\alpha\Psi_i, \Psi_j)_{L^2(\mathbb{R})}\big)_{1\le i,j\le N},$$

where N is an integer taken equal to 50 and $(\Psi_j)_{j\in\mathbb{N}^*}$ stands for the basis of $L^2(\mathbb{R})$ of Hermite functions. Numerical results illustrate the spectral stability in the self-adjoint case. We also notice a strong instability in the non-self-adjoint case, which leads to the computation of 'false eigenvalues' for high energies. In this last case, the resolvent may be very large in norm far from the spectrum.

1.2. Definition of the pseudo-spectra and injectivity pseudo-spectra. — Our interest in this article is to study some notion of semi-classical pseudo-spectrum. In order to justify the definition in the semi-classical setting, we start again with the last example of the rotated harmonic oscillator. Following Zworski in [12],

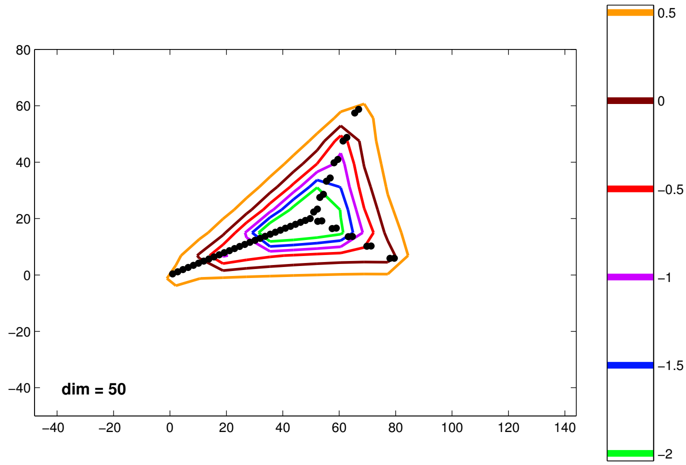

FIGURE 2. Computation of ε-pseudo-spectra for the rotated harmonic oscillator P_α with $\alpha = \frac{\pi}{4}$. The right column gives the corresponding values of $\log_{10}\varepsilon$.

we rephrase the problem of finding eigenvalues for operator P_α by a change of scaling. Setting $y = h^{1/2}x$ where h is a positive parameter, one has

$$P_\alpha - \lambda = -\frac{d^2}{dx^2} + e^{i\alpha}x^2 - \lambda = \frac{1}{h}\Big(- h^2\frac{d^2}{dy^2} + e^{i\alpha}y^2 - h\lambda\Big) = \frac{1}{h}\big(P_\alpha(h) - z\big)$$

where $z = h\lambda$ and

$$P_\alpha(h) = -h^2\frac{d^2}{dy^2} + e^{i\alpha}y^2.$$

Since we are interested in the behaviour of the resolvent for large values of λ, we can work in the semi-classical limit, i.e., $h \to 0$, with z fixed. We can now extend in a natural way the definition of pseudo-spectrum in the semi-classical setting as follows

DEFINITION 1.2.1. — *Let $(P_h)_{0<h\leq 1}$ be a semi-classical family of operators on $L^2(\mathbb{R}^n)$ defined on a domain D, for all $\mu \geq 0$ the set*

$$\Lambda_\mu^{\mathrm{sc}}(P_h) = \{z \in \mathbb{C} : \forall C > 0, \forall h_0 > 0, \exists\, 0 < h < h_0,\ \|(P_h - z)^{-1}\| \geq Ch^{-\mu}\},$$

is called the pseudo-spectrum of index μ of the family $(P_h)_{0<h\leq 1}$ (we write by convention $\|(P_h - z)^{-1}\| = +\infty$ if z belongs to the spectrum of P_h). The pseudo-spectrum of infinite index is defined by

$$\Lambda_\infty^{\mathrm{sc}}(P_h) = \bigcap_{\mu\geq 0}\Lambda_\mu^{\mathrm{sc}}(P_h).$$