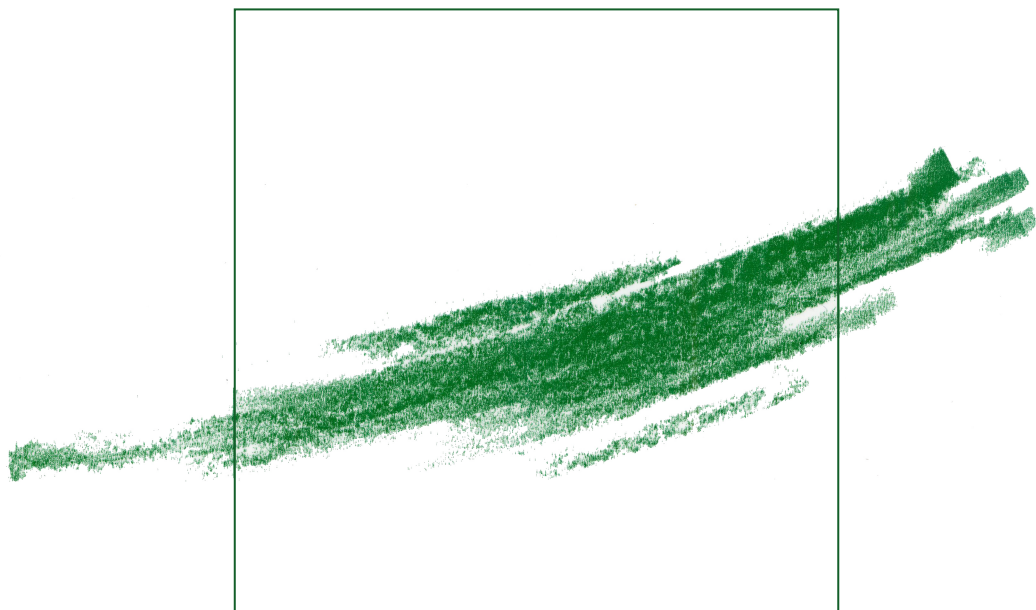


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Integral geometry from Buffon to geometers of today

Rémi LANGEVIN



23

INTEGRAL GEOMETRY FROM BUFFON TO GEOMETERS OF TODAY

Rémi Langevin

Comité de rédaction

Antoine CHAMBERT-LOIR
Julie DÉSERTI

Bertrand MAURY

Grégory MIERMONT (Directeur)

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
smf@smf.univ-mrs.fr

Hindustan Book Agency
O-131, The Shopping Mall
Arjun Marg, DLF Phase 1
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Secrétariat : Nathalie Christiaën

Cours Spécialisés
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
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Rémi Langevin

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