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***Abstract.* —** We consider semi-classical Schrödinger operators with potentials supported in a bounded strictly convex subset $\mathcal{O}$ of $\mathbb{R}^n$ with smooth boundary. Letting h denote the semi-classical parameter, we consider classes of small random perturbations and show that with probability very close to 1, the number of resonances in rectangles $[a, b] - i[0, ch^{\frac{2}{3}}[$, is equal to the number of eigenvalues in $[a, b]$ of the Dirichlet realization of the unperturbed operator in $\mathcal{O}$ up to a small remainder.

***Résumé* (Loi de Weyl pour des résonances semi-classiques associées aux potentiels avec perturbations aléatoires)**

On considère des opérateurs de Schrödinger dont les potentiels ont leur supports dans un ensemble strictement convexe à bord lisse $\mathcal{O} \Subset \mathbb{R}^n$. En désignant par h le paramètre semi-classique, nous considérons des classes de petites perturbations aléatoires et montrons qu'avec une probabilité très proche de 1, le nombre de résonances dans des rectangles $[a, b] - i[0, ch^{\frac{2}{3}}[$ est égal (à un petit reste près) au nombre de valeurs propres dans $[a, b]$ de la réalisation de Dirichlet de l'opérateur dans $\mathcal{O}$.

CONTENTS

CHAPTER 1

INTRODUCTION

There is now a very large literature about the distribution of scattering poles (resonances) often using methods from non-self-adjoint spectral theory and microlocal analysis, including many results about upper and lower bounds on the density of resonances. See for instance [**34**], [**6**] and the references given there. Less is known about actual asymptotics for the number of resonances in various domains. In this paper we shall give such a result for the semi-classical Schrödinger operator

$$P = -h^2\Delta + V(x), \tag{1.1}$$

on $\mathbb{R}^n$ where $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ has compact support.

Recall that the resonances or scattering poles of the operator (1.1) can be defined as the poles of the meromorphic extension of the resolvent

$$(P-z)^{-1} : C_0^\infty(\mathbb{R}^n) \longrightarrow H^2_{\mathrm{loc}}(\mathbb{R}^n)$$

across the positive real axis, to the logarithmic covering space of $\mathbb{C}\setminus\{0\}$ when n is even and to the double covering when n is odd. Alternatively we can continue $(P-k^2)^{-1}$ from the upper half-plane across $\mathbb{R}\setminus\{0\}$ which gives a meromorphic function on $\mathbb{C}$ when n is odd. Using the second definition, we can introduce the number $N(r)$ of resonances in the disc $D(0,r)$ when n is odd.

In one dimension and for $h=1$, M. Zworski [**38**] showed that if $[a,b]$ is the convex hull of the support of V, then

$$N(r) = \frac{2(b-a)}{\pi} r + o(r), \quad r\to\infty, \tag{1.2}$$

which is 2 times the asymptotic number of eigenvalues $\leq r^2$ of the Dirichlet realization of $-\Delta + V$ on $[a,b]$, the factor 2 being explained by the fact that the resonances are symmetric around the imaginary axis. He also showed that most of these concentrate to narrow sectors around the real axis. This extended an earlier result of T. Regge [**20**]. Subsequently, B. Simon [**21**] gave a different proof, inspired by the work of R. Froese [**12**], who got similar results for potentials that do not necessarily have compact support but are very small near infinity. See also the recent works [**8**], [**7**], [**10**] about Weyl and non-Weyl asymptotics for graphs.

In higher odd dimensions, M. Zworski [**40**] considered the case of radial potentials of the form

$$V(x) = f\big(|x|\big)$$

with support in $\overline{B(0,a)}$ where $f \in C^2([0,a])$, $a > 0$, $f(a) \neq 0$ and obtained a Weyl type asympotics (still with $h = 1$),

$$N(r) = K_n a^n r^n + o(r^n), \quad r \to +\infty, \tag{1.3}$$

where $K_n > 0$. Recall also that Zworski [**39**] gave an upper bound in the non-radial case with the correct power of r and using his analysis, P. Stefanov [**34**], gave an explicit formula for the constant $K_n a^n$ in the radial case and showed that the right hand side of (1.3) is up to $o(r^n)$ the sum of 2 times the number of eigenvalues $\leq r^2$ for the interior Dirichlet problem in the ball $B(0,a)$ and the number of scattering poles for the exterior Dirichlet Laplacian in $\mathbb{R}^n \setminus B(0,a)$. (See also G. Vodev [**35**].) He also showed (as a corollary of a more general result for operators with black box) that if we drop the radiality assumption and only assume that $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ has its support in $\overline{B(0,a)}$, then we have the upper bound

$$N(r) \leq K_n a^n r^n + o(r^n), \quad r \to +\infty. \tag{1.4}$$

T. Christiansen [**6**] introduced the set $\mathfrak{M}_a$ of L^∞ potentials V with support in $\overline{B(0,a)}$ for which we have (1.3) and gave the leading asymptotics, of the form Cr^n, for the number of resonances in sectors in the lower half-plane intersected with the disc $D(0,r)$. These formulas were implicit in [**40**], [**34**] in the case of the radial potentials considered there. In particular, when considering smaller and smaller sectors adjacent to $\mathbb{R}_+$ or $\mathbb{R}_-$ we can see, using Lemma 3.3 of [**6**] and some wellknown formulas for the Γ function and the volume of the unit ball, that the constant C converges to the one we get in the leading Weyl asymptotics for the number of Dirichlet eigenvalues for the Laplacian in $B(0,a)$. In the theorems 1.2, 1.3 of the same paper the author gives interesting extensions "for most values of z" to the case of potentials $V(x,z)$ depending holomorphically on a parameter z with $\operatorname{supp} V(\,.\,,z) \subset \overline{B(0,a)}$ such