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SHINICHI MOCHIZUKI

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AN INTRODUCTION TO p-ADIC TEICHMÜLLER THEORY

by

Shinichi Mochizuki

Abstract. — In this article, we survey a theory, developed by the author, concerning the *uniformization of p-adic hyperbolic curves and their moduli.* On the one hand, this theory generalizes the Fuchsian and Bers uniformizations of complex hyperbolic curves and their moduli to nonarchimedean places. It is for this reason that we shall often refer to this theory as *p-adic Teichmüller theory*, for short. On the other hand, this theory may be regarded as a fairly precise hyperbolic analogue of the Serre-Tate theory of ordinary abelian varieties and their moduli.

The central object of p-adic Teichmüller theory is the *moduli stack of nilcurves.* This moduli stack forms a finite flat covering of the moduli stack of hyperbolic curves in positive characteristic. It parametrizes hyperbolic curves equipped with auxiliary "uniformization data in positive characteristic." The geometry of this moduli stack may be *analyzed combinatorially* locally near infinity. On the other hand, a global analysis of its geometry gives rise to a *proof of the irreducibility of the moduli stack of hyperbolic curves using positive characteristic methods.* Various portions of this stack of nilcurves admit *canonical p-adic liftings*, over which one obtains *canonical coordinates and canonical p-adic Galois representations.* These canonical coordinates form the analogue for hyperbolic curves of the canonical coordinates of Serre-Tate theory and the p-adic analogue of the Bers coordinates of Teichmüller theory. Moreover, the resulting Galois representations shed new light on the outer action of the Galois group of a local field on the profinite completion of the Teichmüller group.

1. From the Complex Theory to the "Classical Ordinary" p-adic Theory

In this §, we attempt to bridge the gap for the reader between the classical uniformization of a hyperbolic Riemann surface that one studies in an undergraduate complex analysis course and the point of view espoused in [**21, 22**].

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1.1. The Fuchsian Uniformization. — Let X be a *hyperbolic* algebraic curve over $\mathbb{C}$, the field of complex numbers. By this, we mean that X is obtained by removing r points from a smooth, proper, connected algebraic curve of genus g (over $\mathbb{C}$), where $2g-2+r>0$. We shall refer to (g,r) as the *type* of X. Then it is well-known that to X, one can associate in a natural way a Riemann surface $\mathbf{X}$ whose underlying point set is $X(\mathbb{C})$. We shall refer to Riemann surfaces $\mathbf{X}$ obtained in this way as "hyperbolic of finite type."

Now perhaps the most fundamental *arithmetic* – read "arithmetic at the infinite prime" – fact known about the *algebraic* curve X is that $\mathbf{X}$ admits a uniformization by the upper half plane $\mathbf{H}$:

$$\mathbf{H} \to \mathbf{X}$$

For convenience, we shall refer to this uniformization of $\mathbf{X}$ in the following as the *Fuchsian uniformization of* $\mathbf{X}$. Put another way, the uniformization theorem quoted above asserts that the universal covering space $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ (which itself has the natural structure of a Riemann surface) is holomorphically isomorphic to the upper half plane $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. This fact was "familiar" to many mathematicians as early as the last quarter of the nineteenth century, but was only proven rigorously much later by Koebe.

> *The fundamental thrust of* [**21, 22**] *is to generalize the Fuchsian uniformization to the p-adic context.*

At this point, the reader might be moved to interject: But hasn't this already been achieved decades ago by Mumford in [**25**]? In fact, however, Mumford's construction gives rise to a p-adic analogue *not of the Fuchsian uniformization, but rather of the Schottky uniformization* of a complex hyperbolic curve. Even in the complex case, the Schottky uniformization is an entirely different sort of uniformization – both geometrically and arithmetically – from the Fuchsian uniformization: for instance, its periods are holomorphic, whereas the periods that occur for the Fuchsian uniformization are only real analytic. This phenomenon manifests itself in the nonarchimedean context in the fact that the construction of [**25**] really has nothing to do with a fixed prime number "p," and in fact, takes place entirely in the formal analytic category. In particular, the theory of [**25**] has nothing to do with "Frobenius." By contrast, the theory of [**21, 22**] depends very much on the choice of a prime "p," and makes essential use of the "action of Frobenius." Another difference between the theory of [**25**] and the theory of [**21, 22**] is that [**25**] only addresses the case of curves whose "reduction modulo p" is totally degenerate, whereas the theory of [**21, 22**] applies to curves whose reduction modulo p is only assumed to be "sufficiently generic." Thus, at any rate, the theory of [**21, 22**] is entirely different from and has little directly to do with the theory of [**25**].

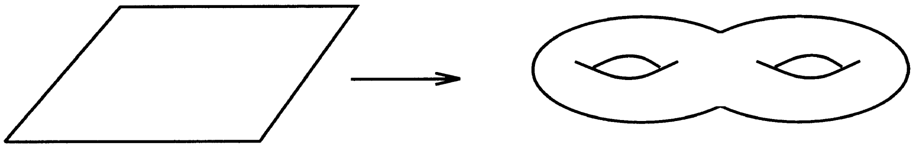

FIGURE 1. The Fuchsian Uniformization

1.2. Reformulation in Terms of Metrics. — Unfortunately, if one sets about trying to generalize the Fuchsian uniformization $\mathbf{H} \to \mathbf{X}$ to the p-adic case in any sort of naive, literal sense, one immediately sees that one runs into a multitude of apparently insurmountable difficulties. Thus, it is natural to attempt to recast the Fuchsian uniformization in a more universal form, a form more amenable to relocation from the archimedean to the nonarchimedean world.

One natural candidate that arises in this context is the notion of a metric – more precisely, the notion of a *real analytic Kähler metric*. For instance, the upper half plane admits a natural such metric, namely, the metric given by

$$\frac{dx^2 + dy^2}{y^2}$$

(where $z = x + iy$ is the standard coordinate on $\mathbf{H}$). Since this metric is invariant with respect to all holomorphic automorphisms of $\mathbf{H}$, it induces a natural metric on $\widetilde{\mathbf{X}} \cong \mathbf{H}$ which is independent of the choice of isomorphism $\widetilde{\mathbf{X}} \cong \mathbf{H}$ and which descends to a metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$.

Having constructed the canonical metric $\mu_{\mathbf{X}}$ on $\mathbf{X}$, we first make the following observation:

There is a general theory of canonical coordinates associated to a real analytic Kähler metric on a complex manifold.

(See, e.g., [**21**], Introduction, § 2, for more technical details.) Moreover, the canonical coordinate associated to the metric $\mu_{\mathbf{X}}$ is precisely the coordinate obtained by pulling back the standard coordinate "z" on the unit disc via any holomorphic isomorphism of $\widetilde{\mathbf{X}} \cong \mathbf{H}$ with the unit disc. Thus, in other words, passing from $\mathbf{H} \to \widetilde{\mathbf{X}}$ to $\mu_{\mathbf{X}}$ is a "faithful operation," i.e., one doesn't really lose any information.

Next, let us make the following observation: Let $\mathcal{M}_{g,r}$ denote the moduli stack of smooth r-pointed algebraic curves of genus g over $\mathbb{C}$. If we order the points that were removed from the compactification of X to form X, then we see that X defines a point $[X] \in \mathcal{M}_{g,r}(\mathbb{C})$. Moreover, it is elementary and well-known that the cotangent space to $\mathcal{M}_{g,r}$ at $[X]$ can be written in terms of square differentials on X. Indeed, if, for simplicity, we restrict ourselves to the case $r = 0$, then this cotangent space is naturally isomorphic to $Q \stackrel{\text{def}}{=} H^0(X, \omega_{X/\mathbb{C}}^{\otimes 2})$ (where $\omega_{X/\mathbb{C}}$ is the algebraic coherent sheaf of differentials on X). Then the observation we would like to make is the following: Reformulating the Fuchsian uniformization in terms of the metric $\mu_{\mathbf{X}}$ allows us to "push-forward" $\mu_{\mathbf{X}}$ to obtain a canonical real analytic Kähler metric $\mu_{\mathbf{M}}$ on the complex analytic stack $\mathbf{M_{g,r}}$ associated to $\mathcal{M}_{g,r}$ by the following formula: if $\theta, \psi \in Q$, then

$$\langle \theta, \psi \rangle \stackrel{\text{def}}{=} \int_{\mathbf{X}} \frac{\theta \cdot \overline{\psi}}{\mu_{\mathbf{X}}}$$

(Here, $\overline{\psi}$ is the complex conjugate differential to ψ, and the integral is well-defined because the integrand is the quotient of a $(2,2)$-form by a $(1,1)$-form, i.e., the integrand is itself a $(1,1)$-form.)

This metric on $\mathbf{M_{g,r}}$ is called the *Weil-Petersson metric.* It is known that

> *The canonical coordinates associated to the Weil-Petersson metric coincide with the so-called Bers coordinates on $\widetilde{\mathbf{M}}_{g,r}$ (the universal covering space of $\mathbf{M_{g,r}}$).*

The Bers coordinates define an anti-holomorphic embedding of $\widetilde{\mathbf{M}}_{g,r}$ into the complex affine space associated to Q. We refer to the Introduction of [**21**] for more details on this circle of ideas.

At any rate, in summary, we see that much that is useful can be obtained from this reformulation in terms of metrics. However, although we shall see later that the reformulation in terms of metrics is not entirely irrelevant to the theory that one ultimately obtains in the p-adic case, nevertheless this reformulation is still not sufficient to allow one to effect the desired translation of the Fuchsian uniformization into an analogous p-adic theory.

1.3. Reformulation in Terms of Indigenous Bundles. — It turns out that the "missing link" necessary to translate the Fuchsian uniformization into an analogous p-adic theory was provided by Gunning ([**13**]) in the form of the notion of an *indigenous bundle.* The basic idea is as follows: First recall that the group $\mathrm{Aut}(\mathbf{H})$ of holomorphic automorphisms of the upper half plane may be identified (by thinking about linear fractional transformations) with $\mathrm{PSL}_2(\mathbb{R})^0$ (where the superscripted "0" denotes the connected component of the identity). Moreover, $\mathrm{PSL}_2(\mathbb{R})^0$ is naturally contained inside $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$. Let $\Pi_{\mathbf{X}}$ denote the (topological) fundamental group of $\mathbf{X}$ (where we ignore the issue of choosing a base-point since this will be irrelevant for what we do). Then since $\Pi_{\mathbf{X}}$ acts naturally on $\widetilde{\mathbf{X}} \cong \mathbf{H}$, we get a natural representation

$$\rho_{\mathbf{X}} : \Pi_{\mathbf{X}} \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$$