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SEMI-LINEAR DIFFRACTION OF CONORMAL WAVES

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Résumé. — Nous étudions la régularité conormale de solutions bornées d'équations semi-linéaires strictement hyperboliques dans des domaines à bord diffractif:

$$Pu = f(x, u) \text{ dans } X,\ u\restriction_{\partial X} = 0,\ u \in L^\infty_{\mathrm{loc}}(X).$$

Si $X_- \subset X$ et X est le domaine d'influence de X_-, nous considérons des solutions u telles que $\operatorname{singsupp}(u) \cap X_- \cap \partial X = \emptyset$; de plus nous supposons que $u\restriction_{X_-}$ est conormale par rapport à une hypersurface caractéristique lisse, le front entrant.

Dans le cas de l'équation linéaire $f \equiv 0$, le support singulier de u est contenu dans la réunion du front entrant et du front réfléchi obtenu par les lois de l'optique géométrique. Ces deux surfaces caractéristiques sont tangentes à l'ensemble des rayons rasants, le lieu des points où les bicaractéristiques entrantes sont tangentes au bord. Dans le cas semi-linéaire, nous démontrons que si de nouvelles singularités apparaissent alors elles apparaissent sur le demi-cône caractéristique au-dessus de l'ensemble des rayons rasants. En fait, le théorème de régularité conormale établi dans cet article est beaucoup plus précis.

Pour illustrer notre propos, nous choisirons pour P l'opérateur des ondes à coefficients constants et pour X le produit de $\mathbb{R}_t$ et de l'extérieur d'un obstacle strictement convexe. Alors $X_- = X \cap \{t < -T\}$. Comme donnée initiale, on pourra prendre une primitive locale de l'onde plane $\delta(t - \langle x, \omega\rangle)$ avec T suffisamment grand. La géométrie de ce problème est figurée sur les schémas 1.1 et 1.2.

Abstract. — We study the conormal regularity of bounded solutions to semi-linear strictly hyperbolic equations on domains with diffractive boundaries:

$$Pu = f(x, u) \text{ in } X,\ u\restriction_{\partial X} = 0,\ u \in L^\infty_{\mathrm{loc}}(X).$$

If $X_- \subset X$ and X is the domain of influence of X_- we consider solutions such that $\operatorname{singsupp}(u) \cap X_- \cap \partial X = \emptyset$ and further suppose that $u\restriction_{X_-}$ is conormal with respect to a smooth characteristic hypersurface, the incoming front.

For the linear equation, $f \equiv 0$, the singular support of u is contained in the incoming front and the reflected front obtained using the rules of geometrical optics; these two characteristic surfaces are tangent at the glancing set, the locus of points at which the incoming bicharactersitics are tangent to the boundary. We prove that in the semi-linear case the only new singularites which may occur appear on the characteristic half-cone over the glancing set. The actual conormal regularity result presented in the paper is considerably more precise.

Our assumptions are best illustrated by taking for P the constant coefficient wave equation with X the product of $\mathbb{R}_t$ and the exterior of a strictly convex obstacle. Then $X_- = X \cap \{t < -T\}$ and for the initial data one can take locally an anti-derivative of the plane wave $\delta(t - \langle x, \omega\rangle)$ with T appropriately large. The geometry of this problem in two space dimensions is shown in Figures 1.1 and 1.2.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The purpose of this paper is to describe the conormal regularity for a class of mixed problems for the semi-linear hyperbolic equations.

The study of C^∞ regularity of solutions to non-linear wave equations has had two main directions: finding estimates on the strength of the anomalous singularities, i.e. those not present in the linear interaction, and obtaining geometric restrictions on the location of singularities. Our work is of the latter type. The strength of singularities for non-linear mixed problems has already been investigated with considerable success in [45, 10, 21, 48]. The estimates on the location of singularities are much finer, so stronger assumptions are needed on the incoming waves or the initial data. The most striking example of this was provided by [2] where it is shown that wave-front set restrictions alone still allow the self-spreading of singularities, making the singular support propagate essentially in the same way as the support of the solution. Thus, in full generality, the location of singularities cannot be related to the original geometry except in a trivial way. A technically more challenging construction of a similar example for gliding mixed problems was then given in [47].

The appropriate class of distributions to consider for the incoming waves or the initial data are the *conormal distributions*, as was first noted in [6]. The conormal distributions appear naturally in the linear theory and are a subclass of the *Lagrangian distributions* motivated by geometrical optics. The interaction of conormal waves for interior problems has been investigated in [40, 32, 7, 9, 3, 42, 34] and the formation of non-linear caustics in [18, 19, 11, 27, 43, 44]. For mixed problems, with only transversal reflections allowed, it was shown in [4, 5] that no anomalous singularities appear. One should also mention that examples of 'new' non-linear singularities were provided at an early stage in [39]: namely, the interaction of three plane waves carrying conormal singularities produces a conic surface of new singularities propagating from the triple interaction point. However, in more complicated settings such as the propagation of the swallowtail or diffraction, where the 'new' cones are expected, no examples have yet been constructed. For the interior problems the methods developed in [20] provide a systematic approach to such constructions. Energy estimates used in the work on the lifespan of solutions to semi-linear hyperbolic equations [15, 22] are also, in essence, of conormal type.

If $\Sigma \subset X$ is a C^∞ hypersurface in a C^∞ manifold X, let $\mathcal{V}(X, \Sigma)$ be the Lie algebra of C^∞ vector fields in X tangent to Σ. The space of distributions of finite L^2-based conormal regularity with respect to Σ is then defined by the stability of regularity under the applications of the elements of $\mathcal{V}(X, \Sigma)$:

$$I_k L^2_{\text{loc}}(X, \Sigma) = \{u \in L^2_{\text{loc}}(X) \; : \; V_1 \cdots V_l u \in L^2_{\text{loc}}(X) \text{ for } \; l \leq k \text{ and } \; V_i \in \mathcal{V}(X, \Sigma)\}.$$

This modifies the definition of the Sobolev space $H_{(k)}$ by placing some geometric restrictions on the differentiations. Nevertheless, as observed in [32], bounded conormal functions have very good multiplicative properties in view of Gagliardo-Nirenberg type inequalities.

Let us now consider a mixed hyperbolic problem with a diffractive boundary (see chapter 2 for a review of definitions). Our object of study is the semi-linear equation:

$$Pu = f(x, u) \text{ in } \; X, \; u|_{\partial X} = 0, \; u|_{X_-} = u_0 \tag{1.1}$$

where f is a C^∞ function of its arguments, P is a strictly hyperbolic operator, X is a C^∞ manifold with the boundary ∂X, $X_- = \{x \in X : \phi(x) < -T\}$ with $\phi \in C^\infty(X)$ a time function for P and the time T fixed.

The initial data is assumed to be conormal to the *incident front* F. The reflection rule of geometrical optics produces the *reflected front* R. With the motivation coming again from the geometrical optics we define the *shadow boundary* on ∂X as

$$\Gamma = \partial X \cap \text{cl}[R \cap F \setminus \partial X].$$

The front obtained from the nonlinear interaction is the *forward half-cone*, S_+, of $P-$bicharacteristics starting on Γ. Let us also denote by D_+ and B_+ the two components of the set of glancing characteristics on S_+. A more detailed discussion of the fronts is presented in chapter 2. Fig. 1.1 shows three different time slices and Fig. 1.2 is a space-time picture. Note that R and F are hypersurfaces with singular boundaries.

The crudest form of our result is

Theorem 1.1. — *Let $u \in L^\infty(X)$ be a bounded solution of* (1.1) *with*

$$u_0 \in I_\infty L^2_{\text{loc}}(X_-; F).$$

Then

$$WF_b(u) \subset {}^bN^*R \cup {}^bN^*F \cup {}^bN^*S_+ \cup {}^bN^*B_+ \cup {}^bN^*D_+ \cup {}^bT^*_\Gamma X \setminus 0$$

We refer the reader to [25] and [14], Sect. 18.3 for the definition of the b-wave front set, WF_b, which reduces to the ordinary WF away from the boundary ∂X. We use the natural map $\jmath : T^*X \setminus 0 \to {}^bT^*X \setminus 0$ (see chapter 4 and the references given above) to define ${}^bN^*\Sigma = \jmath(N^*\Sigma)$.

Theorem 1.1 immediately gives the singular support statement:

Corollary 1.2. — *Under the assumptions of Theorem 1.1*

$$\text{sing supp}\, u \subset F \cup R \cup S_+.$$

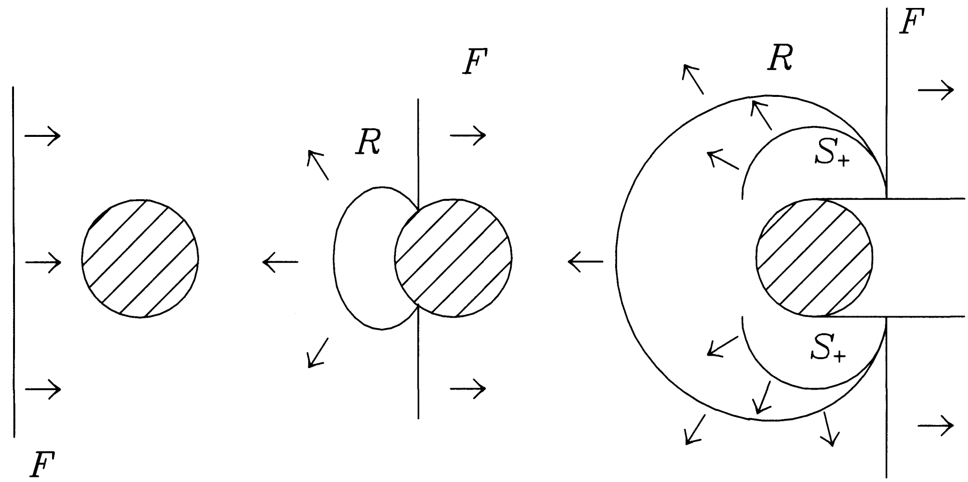

Figure 1.1. The fronts projected to the space variables at fixed times

Since the data u_0 is conormal, one would like to describe precisely the conormal regularity of the solution u. In fact the proof is based on the construction of an appropriate space with good multiplicative and propagative properties – see chapter 3. Since the precise definition of this 'strong', but not quite conormal, space is rather involved we shall content ourselves with a weaker statement here, referring the reader to Definition 3.2 and Theorem 8.2 for the full result.

Theorem 1.3. — *Let $u \in L^\infty_{\mathrm{loc}}(X)$ be a bounded solution of* (1.1) *with*

$$u_0 \in I_k L^2_{\mathrm{loc}}(X_-; F).$$

If Ω is an open subset of X such that

$$\Omega \cap (D_+ \cup B_+) = \emptyset$$

then

$$u|_\Omega \ \in \ I_k L^2_{\mathrm{loc}}(\Omega, F) + I_k L^2_{\mathrm{loc}}(\Omega, R) + I_k L^2_{\mathrm{loc}}(\Omega, S_+).$$

Already in the transversal case this is slightly stronger than the result in [4] as conormal singularities with respect to the boundary are excluded.

Our conclusions are concerned purely with the L^2-based regularity. The present existence theory [45] requires higher Sobolev regularity for u_0 to guarantee local existence of bounded solutions, so one needs to assume $u_0 \in I_k L^2_{\mathrm{loc}}(X_-; F) \cap H_{(s)}(X_-)$ for $s > n/2$. However, the conormal results described above should lead to an improvement in the style of [41]. It should be noted that our present method does not treat the fully semi-linear equation $Pu = f(x, u, \nabla u)$, essentially because the iteration procedure in k proceeds in steps of $1/2$.