AUTOUR DES MOTIFS

École d'été franco-asiatique de géométrie algébrique et de théorie des nombres

Asian-French summer school on algebraic geometry and number theory

Volume I

M. Kim, R. Sujatha,
L. Lafforgue,
A. Genestier, Ngô B. C.

Panoramas et Synthèses

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JULY 17-29 2006

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Volume I

M. Kim, R. Sujatha, L. Lafforgue, A. Genestier, Ngô B. C.

J.-B. Bost et J.-M. Fontaine, éditeurs

Abstract. — This volume contains the first part of the lecture notes of the *Asian-French summer school on algebraic geometry and number theory*, which was held at the Institut des Hautes Études Scientifiques (Bures-sur-Yvette) and the université Paris-Sud XI (Orsay) in July 2006. This summer school was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations.

The contributions in this first part are expanded versions of the talks introducing the theory of motives by M. Kim and R. Sujatha, the lecture notes *Quelques remarques sur le principe de fonctorialité* by L. Lafforgue, and *Lectures on Shimura varieties* by A. Genestier and Ngô B. C.

Résumé. — Ce volume contient la première partie des notes de cours de l'*École d'été franco-asiatique de géométrie algébrique et de théorie des nombres*, qui s'est tenue à l'Institut des Hautes Études Scientifiques et à l'université Paris-Sud XI en juillet 2006. Cette école était consacrée à la théorie des motifs et à ses récents développements, ainsi qu'à des sujets voisins, comme les théories des variétés de Shimura et des représentations automorphes.

Cette première partie est constituée de versions développées des exposés d'introduction à la théorie des motifs par R. Sujatha et M. Kim, puis des notes de cours *Quelques remarques sur le principe de fonctorialité* par L. Lafforgue et *Lectures on Shimura varieties* par A. Genestier et Ngô B. C.

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AVANT-PROPOS

Ce volume contient la première partie des notes de cours de l'école d'été franco-asiatique de géométrie algébrique et de théorie des nombres *Autour des motifs*, qui s'est tenue à l'Institut des Hautes Études Scientifiques et à l'Université Paris-Sud XI en Juillet 2006.

Au nom des membres français du comité d'organisation de l'école d'été, nous voulons exprimer toute notre gratitude aux collègues étrangers qui ont accepté de se joindre à ce comité : leur contribution aux multiples tâches nécessaires à l'organisation d'une telle école — notamment l'obtention de financements, la sélection des étudiants et la mise au point du programme scientifique — a été déterminante.

Nous adressons nos vifs remerciements à Marie-Claude Vergne pour la réalisation de l'affiche de l'école d'été, reproduite en page iv.

L'édition de ce volume a bénéficié des relectures attentives et des suggestions de Laurent Clozel, Laurent Fargues, Fu Lei et Joël Riou. Nous les en remercions chaleureusement.

Orsay, Juillet 2009

Jean-Benoît Bost et Jean-Marc Fontaine

FOREWORD

The *Asian-French Summer School on Algebraic Geometry and Number Theory* was devoted to the theory of motives and its recent developments, and to related topics, notably Shimura varieties and automorphic representations. It was held at the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, and the Université Paris-Sud XI, Orsay, from July 17 to July 29, 2006. The Summer School, although open to all, was more specifically aimed towards doctoral and post-doctoral students. Its purpose was to make accessible some recent progresses in algebraic geometry and number theory, and to stimulate contacts between young researchers from Asia and Europe.

The Organizing Committee consisted of Jean-Benoît Bost, Jean Pierre Bourguignon, Jean-Marc Fontaine, Fu Lei, Laurent Lafforgue, Minhyong Kim, Marc Levine, Ngaiming Mok, Ngô Báo Châu, Takeshi Saito, Sujatha Ramdorai, and Chia-Fu Yu.

Beside the host institutions (IHÉS and Paris-Sud XI), the Summer School benefited from the support of the following organizations: "Arithmetic Algebraic Geometry" European Network, Brain Korea 21, Centre Sino-Français de Mathématiques, Chern Institute of Mathematics (Nankai University), Institut Franco-Indien de Mathématiques, Institut Universitaire de France, Korea Institute for Advanced Study, Japan Society for the Promotion of Science, National Center for Theoretical Sciences (Taiwan), National Science Council (Taiwan), Service pour la Science et la Technologie de l'Ambassade de France en Chine, The Croucher Foundation (Hong Kong).

The program of the Summer School firstly consisted of four series of lectures, scheduled in the morning:

– L. Lafforgue, *Formes automorphes et fonctorialité de Langlands*,
– M. Levine, *Mixed motives and homotopy theory of schemes*,
– Ngô B. C., *Introduction to Shimura varieties*,
– T. Saito, *Galois representations and modular forms.*

In addition, a series of afternoon seminars was organized, devoted to various themes, either supplementing the material discussed in the morning lectures, or providing introductions to some other aspects of the theory of motives. In chronological order, the following topics were discussed during these seminars:

1. *An introduction to "classical" motives*, by M. Kim and R. Sujatha,
2. *Motives and motivic integration for definable sets*, by F. Loeser and R. Cluckers,

3. *Automorphic representations, motives, and Galois representations, I–II–III* by L. Fargues, L. Clozel, P.-H. Chaudouard, and M. Harris,
4. *Quadratic forms and algebraic cycles*, by. Ph. Gille and N. Karpenko,
5. *Periods and motivic Galois groups* and *Motives of non-commutative spaces*, by M. Kontsevich,
6. *Realization functors and the full faithfulness conjectures*, by J. Riou and Y. André,
7. *Complements on categories of motives*, by B. Kahn and F. Ivorra,
8. *Mixed motives and Shimura varieties*, by J. Wildeshaus and D.-C. Cisinsky.

The contributions in this volume are expanded versions of the "introduction" seminars by M. Kim and R. Sujatha and of the lectures by L. Lafforgue and Ngô B. C.

The second volume will contain the notes of the lectures by M. Levine and T. Saito and of seminars 3, 4, 6, 7, 8 above.

Panoramas & Synthèses
29, 2009, p. 1–25

CLASSICAL MOTIVES AND MOTIVIC L-FUNCTIONS

by

Minhyong Kim

The exposition here follows the lecture delivered at the summer school, and hence, contains neither precision, breadth of comprehension, nor depth of insight. The goal rather is the curious one of providing a loose introduction to the excellent introductions that already exist, together with scattered parenthetical commentary. The inadequate nature of the exposition is certainly worst in the third section. As a remedy, the article of Schneider [**39**] is recommended as a good starting point for the complete novice, and that of Nekovář [**36**] might be consulted for more streamlined formalism. For the Bloch-Kato conjectures, the paper of Fontaine and Perrin-Riou [**19**] contains a very systematic treatment, while Kato [**26**] is certainly hard to surpass for inspiration. Kings [**29**], on the other hand, gives a nice summary of results (up to 2003).

1. Motivation

Given a variety X over $\mathbb{Q}$, it is hoped that a suitable analytic function

$$\zeta(X,s),$$

a *ζ-function* of X, encodes important arithmetic invariants of X. The terminology of course stems from the fundamental function

$$\zeta(\mathbb{Q},s)=\sum_{n=1}^{\infty} n^{-s}$$

named by Riemann, which is interpreted in this general context as the zeta function of $\mathrm{Spec}(\mathbb{Q})$. A general zeta function should generalize Riemann's function in a manner similar to Dedekind's extension to number fields. Recall that the latter can be defined by replacing the sum over positive integers by a sum over ideals:

$$\zeta(F,s)=\sum_I N(I)^{-s}$$

where I runs over the non-zero ideals of the ring of integers $\mathcal{O}_F$ and $N(I) = |\mathcal{O}_F/I|$, and that $\zeta(F,s)$ has a simple pole at $s=1$ (corresponding to the trivial motive factor of Spec(F), as it turns out) with

$$(s-1)\zeta(F,s)|_{s=1} = \frac{2^{r_1}(2\pi)^{r_2}h_F R_F}{w_F\sqrt{|D_F|}}$$

By the unique factorization of ideals, $\zeta(F,s)$ can also be written as an Euler product

$$\prod_{\mathcal{P}}(1-N(\mathcal{P})^{-s})^{-1}$$

as $\mathcal{P}$ runs over the maximal ideals of $\mathcal{O}_F$, that is, the closed points of Spec$(\mathcal{O}_F)$. Now, if a scheme $\mathcal{Y}$ is of finite type over $\mathbb{Z}$, then for any closed point $y \in \mathcal{Y}$, its residue field $k(y)$ is finite. Write $N(y) := |k(y)|$. We can then form an Euler product [**40**]

$$Z(\mathcal{Y},s) := \prod_{y\in\mathcal{Y}_0}(1-N(y)^{-s})^{-1},$$

where $(\cdot)_0$ denotes the set of closed points for any scheme $(\cdot)$. In the case when the map

$$\mathcal{Y}\to\mathrm{Spec}(\mathbb{Z})$$

factors through Spec$(\mathbb{F}_p)$, $Z(\mathcal{Y},s)$ reduces to Weil's zeta function for a variety over a finite field (with the substitution $p^{-s}\mapsto t$ if a formal variable has intervened as in [**40**], section 1.6).

When we are starting with $X/\mathbb{Q}$, which we assume throughout to be proper and smooth, a straightforward imitation of Dedekind's definition might involve taking an integral model $\mathcal{X}$ of X, which is a proper flat scheme of finite-type over $\mathbb{Z}$ with X as generic fiber, and defining

$$\zeta(X,s)\text{``}:=\text{''}Z(\mathcal{X},s) = \prod_{x\in\mathcal{X}_0}(1-N(x)^{-s})^{-1}$$

The problem with this approach is that the function thus obtained will depend on the model, and there is no general method for choosing a canonical one. However, there will be some set S of primes such that there is a model $\mathcal{X}_S$ over Spec$(\mathbb{Z}[1/S])$ which is furthermore *smooth*. Even though such a $\mathbb{Z}[1/S]$-model need be no more canonical, it does turn out that the incomplete zeta function

$$\zeta_S(X,s) := \prod_{x\in(\mathcal{X}_S)_0}(1-N(x)^{-s})^{-1}$$

is independent of the model. (More on this point below.) So there are good elementary generalizations of incomplete zeta functions. We note in this connection that

$$Z(\mathcal{X},s) = \prod_p Z(\mathcal{X}_p,s)$$

where

$$\mathcal{X}_p = \mathcal{X}\otimes\mathbb{F}_p$$