ISSN 0012-9593

*quatrième série - tome 50 fascicule 3 mai-juin 2017*

# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique



**Rédaction / *Editor***

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**Tarifs**

Europe : 519 €. Hors Europe : 548 €. Vente au numéro : 77 €.



ISSN 0012-9593

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4$^{e}$ série, t. 50, 2017, p. 503 à 544

# BLOWUP FOR BIHARMONIC NLS

BY Thomas BOULENGER AND Enno LENZMANN

ABSTRACT. – We consider the Cauchy problem for the biharmonic (i.e., fourth-order) NLS with focusing nonlinearity given by

$$i\partial_t u = \Delta^2 u - \mu\Delta u - |u|^{2\sigma}u \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^d,$$

where $0 < \sigma < \infty$ for $d \leqslant 4$ and $0 < \sigma \leqslant 4/(d-4)$ for $d \geqslant 5$; and $\mu \in \mathbb{R}$ is some parameter to include a possible lower-order dispersion. In the mass-supercritical case $\sigma > 4/d$, we prove a general result on finite-time blowup for radial data in $H^2(\mathbb{R}^d)$ in any dimension $d \geqslant 2$. Moreover, we derive a universal upper bound for the blowup rate for suitable $4/d < \sigma < 4/(d-4)$. In the mass-critical case $\sigma = 4/d$, we prove a general blowup result in finite or infinite time for radial data in $H^2(\mathbb{R}^d)$. As a key ingredient, we utilize the time evolution of a nonnegative quantity, which we call the (localized) Riesz bivariance for biharmonic NLS. This construction provides us with a suitable substitute for the variance used for classical NLS problems.

In addition, we prove a radial symmetry result for ground states for the biharmonic NLS, which may be of some value for the related elliptic problem.

RÉSUMÉ. – On considère le problème de Cauchy pour NLS biharmonique (i.e., d'ordre quatre) focalisante définie par

$$i\partial_t u = \Delta^2 u - \mu\Delta u - |u|^{2\sigma}u \quad \text{for } (t,x) \in [0,T) \times \mathbb{R}^d,$$

avec $0 < \sigma < \infty$ pour $d \leqslant 4$ et $0 < \sigma \leqslant 4/(d-4)$ pour $d \geqslant 5$; et $\mu \in \mathbb{R}$ est un paramètre destiné à éventuellement inclure un terme dispersif d'ordre inférieur. Dans le cas sur-critique $\sigma > 4/d$, on prouve un résultat général d'explosion en temps fini pour des données radiales dans $H^2(\mathbb{R}^d)$ en toute dimension $d \geqslant 2$. On déduit par ailleurs une borne supérieure universelle pour la vitesse d'explosion moyennée en temps pour certains indices $4/d < \sigma < 4/(d-4)$. Dans le cas critique $\sigma = 4/d$, on prouve ensuite un résultat général d'explosion en temps fini ou infini, toujours pour des solutions à données radiales $H^2(\mathbb{R}^d)$. On utilise là de façon cruciale l'évolution temporelle d'une quantité positive, que nous baptisons la bivariance (locale) de Riesz pour NLS biharmonique. Cette quantité nous sert de substitut avantageux à la variance classiquement utilisée pour l'étude des problèmes NLS.

On prouve enfin l'existence d'un ground state radial pour NLS biharmonique, qui pourra s'avérer utile pour l'étude du problème elliptique associé.

 doi:10.24033/asens.2326

## 1. Introduction and main results

In this paper, we consider the Cauchy problem for the biharmonic (i.e., fourth-order) NLS with focusing power-type nonlinearity given by

$$\begin{cases} i\partial_t u = \Delta^2 u - \mu \Delta u - |u|^{2\sigma} u, \\ u(0,x) = u_0(x) \in H^2(\mathbb{R}^d), \quad u : [0,T) \times \mathbb{R}^d \to \mathbb{C}, \end{cases} \tag{1.1}$$

where $0 < \sigma < \infty$ for $d \leqslant 4$ and $0 < \sigma \leqslant \frac{4}{d-4}$ for $d \geqslant 5$. Here the parameter $\mu \in \mathbb{R}$ allows us to include a possible lower-order dispersion of classical NLS type.

The biharmonic NLS provides a canonical model for nonlinear Hamiltonian PDEs with dispersion of super-quadratic order. Historically, the study of biharmonic NLS goes back to Karpman and Karpman-Shagalov [21, 22] in the physics literature, followed by the work of Fibich-Ilan-Papanicolaou [15], where the rigorous analysis of these models was initiated. In recent years, a considerable amount of work has been devoted to the study of (1.1). For instance, we refer to the works by Ben-Artzi-Koch-Saut [5] and Pausader [33, 32, 34] on well-posedness and scattering for biharmonic NLS; see also [35, 30, 36].

Despite the fact that problem (1.1) bears a lot of resemblance to the classical NLS, several key questions have been out of scope by rigorous analysis up to now. Here, as a chief open problem addressed in this paper, we mention the existence of blowup solutions for problem (1.1), which has been strongly supported by a series of numerical studies done by Fibich and coworkers [3, 2, 1] for mass-critical and mass-supercritical powers $\sigma \geqslant 4/d$. In the present paper, we shall give an affirmative answer to the existence of blowup solutions for radial data in $H^2(\mathbb{R}^d)$ satisfying criteria that appear natural from known results on blowup for NLS and nonlinear wave equations (NLW). As another main result, we also derive a universal upper bound on the blowup rate in the mass-supercritical case for suitable exponents $\sigma > 4/d$.

Before we turn to the statement of the main results, let us mention some general features of the evolution problem considered in this paper. Similar to the classical NLS, equation (1.1) can be viewed as an infinite-dimensional Hamiltonian system, which enjoys the conservation of mass $M[u]$ and energy $E[u]$ that are given by

$$M[u] = \int_{\mathbb{R}^d} |u|^2 \, dx, \tag{1.2}$$

$$E[u] = \frac{1}{2} \int_{\mathbb{R}^d} |\Delta u|^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{1}{2\sigma+2} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, dx. \tag{1.3}$$

Let us emphasize the fact that (1.1) does not possess any Galilean or Lorentz symmetry in contrast to classical NLS or NLW, respectively. With regard to classification of the criticality level for problem (1.1), let us define the number

$$s_c := \frac{d}{2} - \frac{2}{\sigma}. \tag{1.4}$$

If we suppose for the moment that $\mu = 0$ holds in (1.1), we have the exact scaling invariance so that $u(t,x)$ can be mapped to another solution given by

$$u_\lambda(t,x) = \lambda^{\frac{d}{2} - s_c} u(\lambda^4 t, \lambda x) \quad \text{with } \lambda > 0. \tag{1.5}$$

This rescaling preserves the homogeneous $\dot{H}^{s_c}$-norm of the original solution $u(t)$. Note that $s_c = 2$ corresponds to the endpoint case $\sigma = \frac{4}{d-4}$ in (1.1) for dimensions $d \geqslant 5$. In view of the conservation laws above, we refer to the cases $s_c < 0$, $s_c = 0$, and $s_c > 0$ as *mass-subcritical, mass-critical,* and *mass-supercritical,* respectively. The endpoint case $s_c = 2$ is *energy-critical*.

From [33] we recall the local well-posedness of the Cauchy problem (1.1) holds for $s_c \leqslant 2$. Furthermore, if $s_c < 2$, we have the following blowup alternative: Either the solution $u \in C^0([0,T); H^2(\mathbb{R}^d))$ of (1.1) extends to all times $t \geqslant 0$, or we have that

$$\lim_{t \uparrow T} \|\Delta u(t)\|_{L^2} = +\infty$$

for some finite time $0 < T < +\infty$. In the energy-critical case $s_c = 2$, we have a blowup alternative that involves a critical Strichartz norm in space-time; see Theorem 4 below for more details.

Finally, we mention that, in the mass-subcritical case $s_c < 0$, the conservation laws for $M[u]$ and $E[u]$ together with an interpolation estimate (see (1.6) below) imply that all solutions $u(t)$ of problem (1.1) extend to all times, and thus blowup cannot occur in the mass-subcritical case $s_c < 0$ in analogy to well-posedness theory for classical NLS. The present paper will show that, for $s_c \geqslant 0$, we do have blowup for biharmonic NLS for radial solutions in $H^2$ that satisfy suitable criteria.

### 1.1. Blowup for mass-supercritical case

First, we discuss the case of mass-supercritical powers in (1.1) below the energy-critical level, i.e., we suppose that

$$0 < s_c < 2.$$

In view of the conservation laws for mass and energy, we recall the Gagliardo-Nirenberg (GN) interpolation inequality

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{d,\sigma} \|\Delta u\|_{L^2}^{\frac{\sigma d}{2}} \|u\|_{L^2}^{2-\frac{\sigma}{2}(d-4)} \tag{1.6}$$

valid for all $u \in H^2(\mathbb{R}^d)$ and where $C_{d,\sigma} > 0$ denotes the optimal constant; we refer to Appendix A for more details. It is known that (1.6) has optimizers $Q \in H^2(\mathbb{R}^d)$, which we refer to as *ground states* throughout the following. By rescaling, we can assume that any such ground state $Q \in H^2(\mathbb{R}^d)$ solves the nonlinear elliptic equation

$$\Delta^2 Q + Q - |Q|^{2\sigma} Q = 0 \quad \text{in } \mathbb{R}^d. \tag{1.7}$$

We remark that uniqueness of $Q$ (modulo translation and phase) is not known. In fact, to the best of our knowledge, it has not even been known whether $Q$ can be chosen radially symmetric, since classical methods (e. g., moving planes or rearrangement techniques in $x \in \mathbb{R}^d$) are not applicable for Equation (1.7) due to the presence of the biharmonic operator $\Delta^2$. But if we assume that $\sigma \in \mathbb{N}$ holds, we show that $Q$ can always be chosen to be radially symmetric and real-valued, by using rearrangement techniques in Fourier space; see Appendix A for more details. Actually, we will not make use of this fact shown here. But this symmetry result for ground states $Q$ seems to be new and it is perhaps of some independent value.

Our first main result gives sufficient criteria for finite-time blowup for (1.1) in the class of radial initial data.