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UNIQUENESS OF AXISYMMETRIC VISCOUS FLOWS ORIGINATING FROM CIRCULAR VORTEX FILAMENTS

BY Thierry GALLAY AND Vladimír ŠVERÁK

Abstract. – The incompressible Navier-Stokes equations in $\mathbb{R}^3$ are shown to admit a unique axisymmetric solution without swirl if the initial vorticity is a circular vortex filament with arbitrarily large circulation Reynolds number. The emphasis is on uniqueness, as existence has already been established in [10]. The main difficulty which has to be overcome is that the nonlinear regime for such flows is outside of applicability of standard perturbation theory, even for short times. The solutions we consider are archetypal examples of viscous vortex rings, and can be thought of as axisymmetric analogs of the self-similar Lamb-Oseen vortices in two-dimensional flows. Our method provides the leading term in a fixed-viscosity short-time asymptotic expansion of the solution, and may in principle be extended so as to give a rigorous justification, in the axisymmetric situation, of higher-order formal asymptotic expansions that can be found in the literature [7].

Résumé. – Nous montrons que les équations de Navier-Stokes incompressibles dans $\mathbb{R}^3$ possèdent une unique solution axisymétrique sans swirl lorsque le tourbillon initial est un filament circulaire dont le nombre de Reynolds de circulation peut être arbitrairement grand. L'accent est mis ici sur l'unicité, car l'existence a déjà été établie dans [10]. La difficulté principale à surmonter est que, pour de tels écoulements, le régime non linéaire ne peut être décrit par une théorie de perturbation standard, même pour des temps petits. Les solutions que nous construisons sont des exemples typiques d'anneaux tourbillonnaires visqueux, et peuvent être considérées comme l'analogue axisymétrique des tourbillons autosimilaires de Lamb-Oseen que l'on rencontre dans les écoulements plans. Notre méthode fournit le terme dominant d'un développement asymptotique de la solution à temps petits, la viscosité étant fixée, et peut en principe se généraliser à des ordres plus élevés et donner ainsi une justification complète, dans le cadre axisymétrique, des développements asymptotiques formels que l'on trouve dans la littérature [7].

1. Introduction

In three-dimensional ideal fluids, a *vortex ring* is an axisymmetric flow with the property that the vorticity is entirely concentrated in a solid torus, which moves with constant speed along the symmetry axis. The vortex lines form large circles that fill the torus, whereas

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fluid particles spin around the vortex core within perpendicular cross sections. If $\bar{r}, r$ denote the major and minor radii of the torus, respectively, and if Γ is the flux of the vorticity vector through any cross section, the "local induction approximation" gives the following expression for the translation speed along the axis

$$V = \frac{\Gamma}{4\pi\bar{r}}\Big(\log\frac{1}{\epsilon} + \mathcal{O}(1)\Big), \tag{1.1}$$

which is valid in the asymptotic regime where the aspect ratio $\epsilon = r/\bar{r}$ is small. For the three-dimensional Euler equations, existence of large families of uniformly translating vortex ring solutions has been obtained using fixed point methods [11] or variational techniques [2, 12, 14], and Formula (1.1) has been rigorously justified when $\epsilon \ll 1$ [11, 14]. In addition, for general initial data that are close enough to a vortex ring with small aspect ratio, it is known that the solution evolves in such a way that the vorticity remains sharply concentrated, for a relatively long time, near a vortex ring whose speed is given by (1.1), see [4].

The situation is quite different for viscous fluids, in which uniformly translating vortex rings cannot exist because all localized structures are eventually spread out by diffusion. In that case, however, it is quite natural to consider the initial value problem with a *vortex filament* as initial data, namely a vortex ring with infinitesimal cross section and yet nonzero circulation Γ, so that the initial vorticity is a measure supported by a circle of radius $\bar{r}$. It is then expected that the solution at time $t > 0$ will be close to a vortex ring with Gaussian vorticity profile and minor radius $r = \sqrt{\nu t}$, where ν is the kinematic viscosity. Moreover, this vortex will move along its symmetry axis at a speed given by (1.1), as long as the time-dependent aspect ratio $\epsilon = \sqrt{\nu t}/\bar{r}$ is sufficiently small.

Justifying these heuristic considerations requires some work. For singular initial data such as vortex filaments, the best available results on the Cauchy problem for the three-dimensional Navier-Stokes equations provide existence of a (unique and global) solution only if the circulation parameter Γ is small enough compared to viscosity, see [21, 25]. For larger values of Γ/ν, existence of a (global) axisymmetric solution without swirl has been recently obtained by H. Feng and the second author [10], using approximation techniques that do not give any information about uniqueness, even within the axisymmetric class. In this paper, our main purpose is to fill this gap and to prove that, if one starts from a circular vortex filament with arbitrary strength Γ, the Navier-Stokes equations have a *unique* axisymmetric solution without swirl, which is global and smooth for positive times. This axisymmetric solution is the archetype of a viscous vortex ring, just as the two-dimensional Lamb-Oseen solution is the archetype of a viscous columnar vortex [19]. Our approach is constructive and allows us to determine the leading term in the short-time asymptotic expansion of the vortex ring for a fixed viscosity. In principle, performing the calculations to higher orders in the spirit of Callegari and Ting's paper [7], one should be able to obtain more precise approximations of the solution that remain valid as long as the aspect ratio $\epsilon = \sqrt{\nu t}/\bar{r}$ is small enough. In particular, computing the next order after the leading term, we should recover the asymptotic formula (1.1) for the translation speed if $|\Gamma|/\nu \gg 1$. We leave this extension for future work.

To state our results in a more precise way, we start from the Navier-Stokes equations

$$\partial_t u + (u\cdot\nabla)u = \nu\Delta u - \frac{1}{\rho}\nabla p, \qquad \operatorname{div} u = 0, \tag{1.2}$$

in the whole space $\mathbb{R}^3$, where $u = u(x,t) \in \mathbb{R}^3$ denotes the velocity field and $p = p(x,t) \in \mathbb{R}$ is the internal pressure. Both the kinematic viscosity $\nu > 0$ and the fluid density $\rho > 0$ are assumed to be constant. We restrict ourselves to *axisymmetric solutions without swirl* for which the velocity field u and the vorticity $\omega = \operatorname{curl} u$ have the particular form:

$$u(x,t) = u_r(r,z,t)e_r + u_z(r,z,t)e_z, \qquad \omega(r,z,t) = \omega_\theta(r,z,t)e_\theta. \tag{1.3}$$

Here (r,θ,z) are the usual cylindrical coordinates in $\mathbb{R}^3$, such that $x = (r\cos\theta, r\sin\theta, z)$ for any $x \in \mathbb{R}^3$, and e_r, e_θ, e_z denote the unit vectors in the radial, toroidal, and vertical directions, respectively. The axisymmetric vorticity $\omega_\theta = \partial_z u_r - \partial_r u_z$ satisfies the evolution equation

$$\partial_t\omega_\theta + u\cdot\nabla\omega_\theta - \frac{u_r}{r}\omega_\theta = \nu\Big(\Delta\omega_\theta - \frac{\omega_\theta}{r^2}\Big), \tag{1.4}$$

where $u\cdot\nabla = u_r\partial_r + u_z\partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2$ is the axisymmetric Laplace operator in cylindrical coordinates. The velocity u can be expressed in terms of the axisymmetric vorticity ω_θ by solving the linear elliptic system

$$\partial_r u_r + \frac{1}{r}u_r + \partial_z u_z = 0, \qquad \partial_z u_r - \partial_r u_z = \omega_\theta, \tag{1.5}$$

in the half-plane $\Omega = \{(r,z) \in \mathbb{R}^2 \,|\, r > 0,\ z \in \mathbb{R}\}$. Boundary conditions for the quantities u_r, u_z, and ω_θ are prescribed by requiring that the vectorial functions u, ω in (1.3) be smooth across the symmetry axis $r = 0$. One finds that the radial velocity u_r and the axisymmetric vorticity ω_θ should satisfy the homogeneous Dirichlet condition on $\partial\Omega$, whereas the vertical velocity u_z satisfies the homogeneous Neumann condition.

Since the pioneering work of Ukhovskii and Yudovitch [32], and of Ladyzhenskaya [26], it is well known that the axisymmetric Navier-Stokes equations without swirl are globally well-posed for velocities in (appropriate subspaces of) the energy class, see also [1, 27] for further results in this direction. In the recent work [17], the Cauchy problem for the vorticity equation (1.4) is studied using scale invariant function spaces which emphasize the analogy with the two-dimensional vorticity equation. Following [17], we equip the half-plane Ω with the two-dimensional measure $\mathrm{d}r\,\mathrm{d}z$, as opposed to the three-dimensional measure $r\,\mathrm{d}r\,\mathrm{d}z$ which appears more naturally in cylindrical coordinates. In particular, for any $p \in [1,\infty)$, we denote by $L^p(\Omega)$ the space of measurable functions $\omega_\theta : \Omega \to \mathbb{R}$ such that

$$\|\omega_\theta\|_{L^p(\Omega)} := \Big(\int_\Omega |\omega_\theta(r,z)|^p\,\mathrm{d}r\,\mathrm{d}z\Big)^{1/p} < \infty.$$

As usual, the limiting space $L^\infty(\Omega)$ is equipped with the essential supremum norm. We also denote by $\mathcal{M}(\Omega)$ the set of all real-valued finite regular measures on Ω, equipped with the total variation norm

$$\|\mu\|_{\mathrm{tv}} = \sup\Big\{\int_\Omega \phi\,\mathrm{d}\mu \,\Big|\, \phi \in C_0(\Omega),\ \|\phi\|_{L^\infty(\Omega)} \le 1\Big\},$$

where $C_0(\Omega)$ denotes the set of all real-valued continuous functions on Ω that vanish at infinity and on the boundary $\partial\Omega$. Clearly $L^1(\Omega)$ is a closed subspace of $\mathcal{M}(\Omega)$, and $\|\mu\|_{\mathrm{tv}} = \|\omega_\theta\|_{L^1(\Omega)}$ if $\mu = \omega_\theta\,\mathrm{d}r\,\mathrm{d}z$ for some $\omega_\theta \in L^1(\Omega)$.

As is proved in [17, Theorem 1.3], the Cauchy problem for the axisymmetric vorticity equation (1.4) is globally well-posed if the initial vorticity $\mu \in \mathcal{M}(\Omega)$ is a finite measure