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FLAT LINE BUNDLES AND THE CAPPELL-MILLER TORSION IN ARAKELOV GEOMETRY

BY Gerard FREIXAS I MONTPLET
AND Richard A. WENTWORTH

À Jean-Michel Bismut, à l'occasion de son 70^{e} anniversaire, avec admiration.

Abstract. – In this paper, we extend Deligne's functorial Riemann-Roch isomorphism for Hermitian holomorphic line bundles on Riemann surfaces to the case of flat, not necessarily unitary connections. The Quillen metric and $\star$-product of Gillet-Soulé are replaced with complex valued logarithms. On the determinant of cohomology side, we show that the Cappell-Miller torsion is the appropriate counterpart of the Quillen metric. On the Deligne pairing side, the logarithm is a refinement of the intersection connections considered in a previous work. The construction naturally leads to an Arakelov theory for flat line bundles on arithmetic surfaces and produces arithmetic intersection numbers valued in $\mathbb{C}/\pi i\,\mathbb{Z}$. In this context we prove an arithmetic Riemann-Roch theorem. This realizes a program proposed by Cappell-Miller to show that their holomorphic torsion exhibits properties similar to those of the Quillen metric proved by Bismut, Gillet and Soulé. Finally, we give examples that clarify the kind of invariants that the formalism captures; namely, periods of differential forms.

Résumé. – Dans cet article nous étendons l'isomorphisme de Riemann-Roch fonctoriel pour les fibrés en droites holomorphes Hermitiens, dû à Deligne, au cas des fibrés plats non nécessairement unitaires. La métrique de Quillen et le produit $\star$ de Gillet-Soulé sont remplacés par des logarithmes à valeurs complexes. Sur le déterminant de la cohomologie, nous montrons que la torsion de Cappell-Miller est l'analogue approprié de la métrique de Quillen. Sur les accouplements de Deligne, les logarithmes raffinent les connexions d'intersection introduites dans un travail précédent. La construction conduit naturellement à une théorie d'Arakelov pour les fibrés plats sur les surfaces arithmétiques, et produit des nombres d'intersection arithmétique à valeurs dans $\mathbb{C}/\pi i\,\mathbb{Z}$. Dans ce contexte, nous démontrons une formule de Riemann-Roch arithmétique. On réalise ainsi un programme proposé par Cappell-Miller visant à montrer que leur torsion holomorphe possède des propriétés analogues à celles de la métrique de Quillen établies par Bismut, Gillet et Soulé. Finalement, nous donnons des exemples qui clarifient le type d'invariants que ce formalisme encode: des périodes de formes différentielles.

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1. Introduction

Arithmetic intersection theory was initiated by Arakelov [1] in an attempt to approach the Mordell conjecture on rational points of projective curves over number fields by mimicking the successful arguments of the function field case. The new insight was the realization that an intersection theory on arithmetic surfaces could be defined by adding some archimedean information to divisors. This archimedean datum consists of the Green's functions that arise from smooth Hermitian metrics on holomorphic line bundles. The use of a metric structure is also natural for diophantine purposes, as one may want to measure the size of integral sections of a line bundle on an arithmetic surface.

Arakelov's foundational work was complemented by Faltings, who proved among other things, the first version of an arithmetic Riemann-Roch type formula [13]. Later, in a long collaboration starting with [15], Gillet and Soulé vastly extended the theory both to higher dimensions and to more general structures on the archimedean side. Their point of view is an elaboration of the ideas of Arakelov and is cast as a suitable "completion" of the usual Chow groups of classical intersection theory over a Dedekind domain. Their formalism includes arithmetic analogs of characteristic classes of Hermitian holomorphic vector bundles [16, 17]. This led them to develop and prove a general Grothendieck-Riemann-Roch type theorem in this setting [18]. A key ingredient is the *analytic torsion* of the Dolbeault complex associated to a Hermitian holomorphic vector bundle over a compact Kähler manifold. Their proof requires deep properties of the analytic torsion due to Bismut and collaborators [2, 3, 4, 5, 6, 7]. In [12], Deligne proposed a program to lift the Grothendieck-Riemann-Roch theorem to a functorial isomorphism between line bundles that becomes an isometry when the vector bundles are endowed with suitable metrics. This goal was achieved in the case of families of curves. He established a canonical isometry between the determinant of cohomology of a Hermitian vector bundle with the Quillen metric and some Hermitian intersection bundles involving, in particular, the *Deligne pairings* of line bundles.

In our previous work [27], we produced natural connections on Deligne pairings of line bundles with flat relative connections on families of compact Riemann surfaces. These were called *intersection connections*, and they reduce to Deligne's constructions in the case where the relative connections are the Chern connections for a Hermitian structure. As in the case of Deligne's formulation, intersection connections are functorial, and via the Chern-Weil expression they realize a natural cohomological relationship for Deligne pairings. Moreover, we showed that in the case of a trivial family of curves, *i.e.,* a single Riemann surface and a holomorphic family of flat line bundles on it, we could interpret Fay's holomorphic extension of analytic torsion for flat unitary line bundles [14] as the construction of a Quillen type holomorphic connection on the determinant of cohomology. This can be recast as a statement that the Deligne-Riemann-Roch type isomorphism is flat with respect to these connections. The relevant contents of [27] are summarized in Section 2 below.

The results in [27] on intersection and Quillen type connections are vacuous for a single Riemann surface and a single flat holomorphic line bundle, since there are no interesting connections over points! To proceed further, and especially with applications to Arakelov theory in mind, we establish "integrated" versions of the aforementioned connections. The nature of such an object is what we have referred to above as a logarithm of a line bundle $\mathcal{L} \to S$ over a smooth variety S. This takes the place of the logarithm of a Hermitian metric in the classical situation. More precisely, a logarithm is an equivariant map $\mathrm{LOG} : \mathcal{L}^{\times} \to \mathbb{C}/2\pi i\,\mathbb{Z}$. It has an associated connection which generalizes the Chern connection of a Hermitian metric, but which is not necessarily unitary for some Hermitian structure. Although the notion of a logarithm is equivalent simply to a trivialization of the $\mathbb{G}_m$-torsor $\mathcal{L}^{\times}$, it nevertheless plays an important role in the archimedean part of the arithmetic intersection product, as we explain below.

1.1. Quillen-Cappell-Miller and intersection logarithms

Let (X, p) be a compact Riemann surface with a base point, $\overline{X}$ the conjugate Riemann surface, and $\mathcal{L}_\chi \to X$, $\mathcal{L}_\chi^c \to \overline{X}$ rigidified (at p) flat complex line bundles with respective holonomies χ^{-1} and χ, for some character $\chi\colon \pi_1(X, p) \to \mathbb{C}^{\times}$. Applied to these data, Deligne's canonical (up to sign) isomorphism for $\mathcal{L}_\chi$ and $\mathcal{L}_\chi^c$ gives

$$(1)\qquad \mathcal{D}\colon \{\lambda(\mathcal{L}_\chi - \mathcal{O}_X) \otimes_{\mathbb{C}} \lambda(\mathcal{L}_\chi^c - \mathcal{O}_{\overline{X}})\}^{\otimes 2} \xrightarrow{\sim} \langle \mathcal{L}_\chi, \mathcal{L}_\chi \otimes \omega_X^{-1}\rangle \otimes_{\mathbb{C}} \langle \mathcal{L}_\chi^c, \mathcal{L}_\chi^c \otimes \omega_{\overline{X}}^{-1}\rangle,$$

where λ denotes the determinant of coherent cohomology and $\langle\ ,\ \rangle$ denotes the Deligne pairing (see Section 2 below for a review of Deligne's isomorphism). After choosing a metric on T_X, a construction of Cappell-Miller [11] produces a trivialization of the product of determinants of cohomologies, and hence gives rise to a logarithm denoted LOG_Q. For unitary characters, the Cappell-Miller trivialization is equivalent to the Quillen metric. We call LOG_Q the *Quillen-Cappell-Miller logarithm*. Regarding the right hand side of (1), we shall show in Section 4 that the intersection connection of [27] can be integrated to an *intersection logarithm* LOG_{int}. The first main result is the following (see Theorem 5.10):

THEOREM 1.1 (Deligne Isomorphism). – *The map* (1) *is compatible with* LOG_Q *and* LOG_{int}*, modulo* $\pi i\,\mathbb{Z}$*. That is,*

$$(2)\qquad \mathrm{LOG}_Q = \mathrm{LOG}_{int} \circ \mathcal{D}$$

in $\mathbb{C}/\pi i\mathbb{Z}$*.*

The idea of the proof is to deform the line bundles to the universal family over the Betti moduli space $M_B(X) = \mathrm{Hom}(\pi_1(X, p), \mathbb{C}^{\times})$. Over $M_B(X)$, both LOG_Q and LOG_{int} turn out to be holomorphic. Moreover, through Deligne's isomorphism, they agree along the totally real subvariety consisting of unitary characters. This forces the coincidence everywhere. There is however a sign ambiguity, due to the sign ambiguity of Deligne's isomorphism. This explains the equality modulo $i\pi\mathbb{Z}$ instead of $2\pi i\mathbb{Z}$.

The holomorphic behavior of LOG_Q over $M_B(X)$ is established in Section 5. We follow the method of Bismut-Gillet-Soulé [6], which shows that Bismut-Freed's connection [2, 3] is the Chern connection of the Quillen metric with respect to the holomorphic structure on the determinant of cohomology given by the Knudsen-Mumford construction [23]. However, these authors work with Hermitian vector bundles and self-adjoint Laplace type operators. Since the operators here are not self-adjoint their arguments do not directly apply. The presentation below exhibits a holomorphic dependence with respect to parameters in $M_B(X)$. In this context, Kato's theory of analytic perturbations of closed operators [21,