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THE MIYAOKA-YAU INEQUALITY AND UNIFORMISATION OF CANONICAL MODELS

BY DANIEL GREB, STEFAN KEBEKUS, THOMAS PETERNELL AND BEHROUZ TAJI

ABSTRACT. – We establish the Miyaoka-Yau inequality in terms of orbifold Chern classes for the tangent sheaf of any complex projective variety of general type with klt singularities and nef canonical divisor. In case equality is attained for a variety with at worst terminal singularities, we prove that the associated canonical model is the quotient of the unit ball by a discrete group action.

RÉSUMÉ. – Nous établissons l'inégalité de Miyaoka-Yau en termes de classes de Chern orbifoldes pour le faisceau tangent d'une variété complexe projective de type général à singularités klt et diviseur canonique nef. Dans le cas d'égalité pour une variété à singularités terminales, nous établissons que le modèle canonique associé est un quotient de la boule unité par un groupe agissant discrètement.

1. Introduction

A classical result in complex geometry asserts that the Chern classes of any holomorphic, slope-semistable vector bundle $\mathscr{E}$ of rank r on a compact Kähler manifold (X, ω) satisfy the *Bogomolov-Gieseker inequality*

$$\int_X \bigl(2r \cdot c_2(\mathscr{E}) - (r-1) \cdot c_1^2(\mathscr{E})\bigr) \wedge \omega^{n-2} \geq 0.$$

Thanks to his solution of the Calabi conjecture, Yau established in [66] the following stronger, *Miyaoka-Yau inequality* for the holomorphic tangent bundle of any n-dimensional compact Kähler manifold X with ample canonical class K_X,

$$(*) \qquad \int_X \bigl(2(n+1) \cdot c_2(\mathscr{T}_X) - n \cdot c_1^2(\mathscr{T}_X)\bigr) \cdot [K_X]^{n-2} \geq 0.$$

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In case of equality, the natural symmetries imposed by the Kähler-Einstein condition lead to the uniformisation of X by the unit ball.

A fundamental result of Birkar, Cascini, Hacon and McKernan, [4], states that every projective manifold of general type admits a minimal model, which is a normal, $\mathbb{Q}$-factorial, projective variety with at most terminal singularities whose canonical divisor is big and nef. These varieties are however usually singular. It was expected that the Miyaoka-Yau inequality should also hold in this context, with applications to uniformisation in case of equality. This problem has attracted considerable interest; Section 1.5 gives a short account of the history.

1.1. Main results of this paper

The main result of this paper settles the problem in full generality, even in the broader context of varieties with Kawamata log-terminal (= klt) singularities and nef canonical divisor.

THEOREM 1.1 ($\mathbb{Q}$-Miyaoka-Yau inequality). – *Let X be an n-dimensional, projective, klt variety of general type whose canonical divisor K_X is nef. Then,*

$$\Big(2(n+1)\cdot \widehat{c}_2(\mathscr{T}_X) - n\cdot \widehat{c}_1(\mathscr{T}_X)^2\Big)\cdot [K_X]^{n-2} \geq 0. \tag{1.1.1}$$

The formulation of Theorem 1.1 uses the fact that varieties with klt singularities have quotient singularities in codimension two. This allows to define $\mathbb{Q}$-Chern classes (or "orbifold Chern classes") $\widehat{c}_1(\mathscr{T}_X)$ and $\widehat{c}_2(\mathscr{T}_X)$ for the tangent sheaf $\mathscr{T}_X = (\Omega^1_X)^*$ of X, which is reflexive and a $\mathbb{Q}$-vector bundle on the open subset where X has quotient singularities. We refer to Section 3.7 for definitions and for a detailed discussion. If X is smooth in codimension two, which is the case when X has terminal singularities, these agree with the usual Chern classes $c_\bullet(\mathscr{T}_X)$. We call a projective variety of general type *minimal* if it has at worst terminal singularities and if its canonical divisor is nef, cf. [41, 2.13] and Definition 2.3 below.

THEOREM 1.2 (Characterisation of singular ball quotients, I). – *Let X be an n-dimensional minimal variety of general type. If equality holds in* (1.1.1)*, then the canonical model X_{can} is smooth in codimension two, there exists a ball quotient Y and a finite, Galois, quasi-étale morphism $f : Y \to X_{\text{can}}$. In particular, X_{can} has only quotient singularities.*

We refer to Section 2.2 for a discussion of ball quotients and canonical models.

We expect that Theorem 1.2 holds without the additional assumption that X be terminal. In fact, we prove a result slightly stronger than Theorem 1.2, which applies to varieties with klt singularities that are smooth in codimension two, cf. Theorem 8.1 as well as Theorem and Definition 1.3 below. As already said above, Theorem 1.2 applies to all minimal models of smooth varieties of general type, which is the case most relevant for applications in the Minimal Model Program. In Section 10, we discuss further potential generalizations of the Miyaoka-Yau inequality and the uniformisation result.

Extending Theorem 1.2, we show that the canonical models of Theorem 1.2 admit a "singular uniformisation" by the unit ball $\mathbb{B}^n$. More precisely, they can be realized as quotients of $\mathbb{B}^n$ by actions of discrete subgroups in $\mathrm{PSU}(1,n)$ that are not necessarily fixed-point free. In particular, the geometry of these spaces can be studied using the theory of automorphic forms, cf. [40, Part II].

THEOREM AND DEFINITION 1.3 (Characterisation of singular ball quotients, II).

Let X be a normal, irreducible, compact, complex space of dimension n. Then the following statements are equivalent.

(1.3.1) *The space X is of the form $\mathbb{B}^n/\widehat{\Gamma}$ for a discrete, cocompact subgroup $\widehat{\Gamma} < \mathrm{Aut}_{\mathscr{O}}(\mathbb{B}^n)$ whose action on $\mathbb{B}^n$ is fixed-point free in codimension two.*

(1.3.2) *The space X is of the form Y/G, where Y is a ball quotient (cf. Definition 2.5), and G is a finite group of automorphisms of Y whose action is fixed-point free in codimension two.*

(1.3.3) *The space X is projective, klt, and smooth in codimension two; the canonical divisor K_X is ample and we have equality in the $\mathbb{Q}$-Miyaoka-Yau Inequality* (1.1.1).

A compact complex space is called singular ball quotient *if it satisfies these equivalent conditions.*

COROLLARY 1.4 (Hyperbolicity of smooth loci of canonical models).

In the setting of Theorem 1.2, the canonical model X_{can} is a singular ball quotient. In particular, the smooth locus of X_{can} is Kobayashi-hyperbolic.

In fact, a more precise hyperbolicity statement holds, see Section 9.3. In addition, classical results concerning deformation rigidity [5], Mostow rigidity [66, Thm. 6], stability under Galois conjugation [56, Cor. 9.5], and the fact that ball quotients can be defined over number fields [54] have analogs for singular ball quotients. These aspects will be addressed in a future work.

1.2. Outline of the proof

Various earlier papers used differential-geometric techniques, such as orbifold Kähler-Einstein metrics, to obtain the Miyaoka-Yau inequality. Inspired by the work of Simpson [56] we take a different approach, partially generalizing Simpson's results on the Kobayashi-Hitchin correspondence for Higgs sheaves. For suitable manifolds X, Simpson equips $\mathscr{E} := \Omega^1_X \oplus \mathscr{O}_X$ with a natural structure of a Higgs bundle, proves its stability and derives a Bogomolov-Gieseker inequality for $\mathscr{E}$. The Miyaoka-Yau inequality for $\mathscr{T}_X$ is an immediate consequence. In case of equality, he constructs a variation of Hodge structures whose period map gives the desired uniformisation by the ball.

On a technical level, one main contribution of our paper is to establish a good definition of Higgs sheaves on singular spaces, and an associated notion of stability. These definitions may seem a little awkward at first, but for varieties with the singularities of the minimal model program they have just enough universal properties to make Simpson's approach work—the list of properties includes restrictions theorems of Mehta-Ramanathan type, weakly functorial pull-back, and invariance of stability under resolution. As for a converse, earlier work on differential forms, [16, 32], suggests that spaces with klt singularities are the largest