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Matthew MORROW

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

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K-THEORY AND LOGARITHMIC HODGE-WITT SHEAVES OF FORMAL SCHEMES IN CHARACTERISTIC p

BY MATTHEW MORROW

ABSTRACT. – We describe the mod p^r pro K-groups $\{K_n(A/I^s)/p^r\}_s$ of a regular local $\mathbb{F}_p$-algebra A modulo powers of a suitable ideal I, in terms of logarithmic Hodge-Witt groups, by proving pro analogues of the theorems of Geisser-Levine and Bloch-Kato-Gabber. This is achieved by combining the pro Hochschild-Kostant-Rosenberg theorem in topological cyclic homology with the development of the theory of de Rham-Witt complexes and logarithmic Hodge-Witt sheaves on formal schemes in characteristic p.

Applications include the following: the infinitesimal part of the weak Lefschetz conjecture for Chow groups; a p-adic version of Kato-Saito's conjecture that their Zariski and Nisnevich higher dimensional class groups are isomorphic; continuity results in K-theory; and criteria, in terms of integral or torsion étale-motivic cycle classes, for algebraic cycles on formal schemes to admit infinitesimal deformations.

Moreover, in the case $n = 1$, we compare the étale cohomology of $W_r\Omega^1_{\log}$ and the fppf cohomology of μ_{p^r} on a formal scheme, and thus present equivalent conditions for line bundles to deform in terms of their classes in either of these cohomologies.

RÉSUMÉ. – Nous décrivons les K-groups $\{K_n(A/I^s)/p^r\}_s$ modulo p^r d'une $\mathbb{F}_p$-algèbre régulière locale A modulo les puissances d'un idéal approprié I en termes des groupes de Hodge-Witt logarithmique, en démontrant des analogues pro des théorèmes de Geisser-Levine et Bloch-Kato-Gabber. Ceci est accompli en utilisant le théorème d'Hochschild-Kostant-Rosenberg pro en homologie cyclique topologique et le développement de la théorie des complexes de de Rham-Witt et de Hodge-Witt logarithmique sur les $\mathbb{F}_p$-schémas formels.

Des applications incluent les suivants : la partie infinitésimale de la conjecture de Lefschetz faible pour les groupes de Chow ; une version p-adique de la conjecture de Kato-Saito que leurs groupes des classes de dimension supérieure Zariski et Nisnevich sont isomorphes ; des résultats de continuité en K-théorie ; et des conditions, en termes des classes de cycles motiviques étales entières ou torsions, pour que les cycles algébriques sur un schéma formel admettent des déformations infinitésimales.

De plus, dans le cas où $n = 1$ nous comparons la cohomologie étale de $W_r\Omega^1_{\log}$ et la cohomologie fppf de μ_{p^r} sur un schéma formel, et ainsi présentons des conditions équivalentes pour que les fibres en droites déforment en termes de leurs classes dans chacune de ces cohomologies.

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0. Introduction

0.1. K-theory

The primary goal of this article is to extend results concerning the K-theory and motivic cohomology of smooth varieties in characteristic p to the case of regular formal schemes. If A is an $\mathbb{F}_p$-algebra, then we consider the natural homomorphisms

$$(1) \qquad K_n(A)/p^r \longleftarrow K_n^M(A)/p^r \xrightarrow{\text{dlog}[\cdot]} W_r\Omega^n_{A,\log},$$

where $W_r\Omega^n_{A,\log}$ (also denoted by $\nu_r^n(A)$ in the literature) is the subgroup of the Hodge-Witt group $W_r\Omega^n_A$ consisting of elements which can be written étale locally as sums of dlog forms, and the map dlog$[\cdot]$ is given by $\{a_1,\dots,a_n\} \mapsto \text{dlog}[a_1]\cdots\text{dlog}[a_n]$ as usual. If A is regular and local then both of these homomorphisms are known to be isomorphisms: this reduces, via Gersten sequences, to the case where A is a field, in which case the leftwards isomorphism is due to Geisser and Levine [12], who also proved that $K_n(A)$ is p-torsion-free, and the rightwards isomorphism is the Bloch-Kato-Gabber theorem (see Theorem 5.1 for more details and references; also, to avoid issues caused by finite residue fields, we use Kerz-Gabber's improved Milnor K-theory throughout).

We extend these results to the pro abelian groups $\{K_n(A/I^s)\}_s$, where $I \subseteq A$ is an ideal. We must first describe two hypotheses: the first of these is that A is F-finite (i.e., a finitely generated module over its subring of p-th powers); the second is that our closed subschemes Y are often required to be *generalized normal crossing*, or *gnc*, meaning that Y admits a closed cover by subschemes such that the reduced subscheme of any possible multiple intersection is regular (e.g., it suffices for Y to be regular, or for it to be a normal crossing divisor on a regular scheme, which we believe cover all cases of interest for the applications; see Section 1.5); a ring is said to be gnc if and only if its spectrum is.

The following is our pro version of the isomorphisms recalled above:

THEOREM 0.1 (See Thm. 5.4 & Corol. 5.5). – *Let A be a regular, F-finite $\mathbb{F}_p$-algebra, and $I \subseteq A$ an ideal such that A/I is gnc and local; fix $n, r \geq 0$. Then the natural homomorphisms of pro abelian groups*

$$\{K_n(A/I^s)/p^r\}_s \longleftarrow \{K_n^M(A/I^s)/p^r\}_s \xrightarrow{\text{dlog}[\cdot]} \{W_r\Omega^n_{A/I^s,\log}\}_s$$

are surjective and have the same kernel, thereby inducing an isomorphism

$$\{K_n(A/I^s)/p^r\}_s \xrightarrow{\simeq} \{W_r\Omega^n_{A/I^s,\log}\}_s.$$

Moreover, the pro abelian group $\{K_n(A/I^s)\}_s$ is p-torsion-free.

The kernel of the surjection $\{K_n^M(A/I^s)/p^r\}_s \to \{K_n(A/I^s)/p^r\}_s$ appearing in the statement of the theorem is reasonably well controlled; see Section 6.1 for some precise results, where we show in particular that it vanishes if I is principal and A/I is regular. This covers the traditional case of curves on K-theory, namely when $A = R[[t]]$ and $I = (t)$; a consequence of this is a curious, and seemingly new, log/exp isomorphism between the relative part of $W_r\Omega^n_{R[t]/t^s,\log}$ and the big Hodge-Witt groups of R itself:

COROLLARY 0.2 (See Corol. 6.7). – *Let R be a regular, local, F-finite $\mathbb{F}_p$-algebra; fix $n \geq 0$, $r \geq 1$. Then there exists a short exact sequence of pro abelian groups*

$$0 \longrightarrow \{\mathbb{W}_s\Omega^{n-1}_R/p^r\}_s \xrightarrow{\mathrm{dlog}[\cdot]\circ\gamma_n} \{W_r\Omega^n_{R[t]/t^s,\log}\}_s \longrightarrow W_r\Omega^n_{R,\log} \longrightarrow 0,$$

where $\gamma_n : \{\mathbb{W}_{s-1}\Omega^{n-1}_R\}_s \xrightarrow{\simeq} \{K^{\mathrm{sym}}_n(R[t]/t^s,(t))\}_s$ is the original comparison map of Bloch-Deligne-Illusie between the de Rham-Witt complex and curves on K-theory (see Section 6.2 for more details).

Applying $\varprojlim_s$ to Theorem 0.1, together with a continuity result for logarithmic Hodge-Witt groups, we establish the following continuity result for K-theory; this is already known if A/I is regular, thanks to Geisser and Hesselholt [10]:

THEOREM 0.3 (See Thm. 6.9). – *With notation as in Theorem 0.1, assume moreover that A is I-adically complete. Then the canonical maps*

$$K_n(A;\mathbb{Z}/p^r) \longrightarrow \pi_n \operatorname{holim}_s K(A/I^s;\mathbb{Z}/p^r) \longrightarrow \varprojlim_s K_n(A/I^s;\mathbb{Z}/p^r)$$

are isomorphisms for all $n \geq 0$, $r \geq 1$.

We present some similar continuity results for Milnor K-theory in Corollary 6.10.

0.2. Infinitesimal deformations of Chow groups

We prove variations on Theorem 0.1 for relative K-groups and in the context of sheaves in the Zariski, Nisnevich, and étale topologies; combining these with our development of the theory of logarithmic Hodge-Witt groups on formal $\mathbb{F}_p$-schemes, we prove a number of theorems concerning Chow groups and infinitesimal thickenings, including the following:

THEOREM 0.4 (See Thm. 6.11). – *Let X be a regular, F-finite $\mathbb{F}_p$-scheme, and $Y \hookrightarrow X$ a gnc closed subscheme. Then the canonical map of pro abelian groups*

$$\{H^i_{\mathrm{Zar}}(X, \mathscr{K}_{n,(X,Y_s)}/p^r)\}_s \longrightarrow \{H^i_{\mathrm{Nis}}(X, \mathscr{K}_{n,(X,Y_s),\mathrm{Nis}}/p^r)\}_s$$

is an isomorphism for all $n, i, r \geq 0$, where Y_s denotes the s^{th} infinitesimal thickening of Y inside X.

In particular, if X is a smooth variety over a perfect field of characteristic p and $Y \hookrightarrow X$ is a normal crossing divisor, then Theorem 0.4 implies that

$$\varprojlim_s H^i_{\mathrm{Zar}}(X, \mathscr{K}_{n,(X,Y_s)}/p^r) \xrightarrow{\simeq} \varprojlim_s H^i_{\mathrm{Nis}}(X, \mathscr{K}_{n,(X,Y_s),\mathrm{Nis}}/p^r).$$

Replacing Quillen by Milnor K-theory and removing the mod p^r, this was conjectured to be true by Kato and Saito [29, pg. 256] when $i = n = \dim X$, as part of their higher dimensional class field theory, in which the left and right sides play the role of certain Zariski/Nisnevich class groups in their theory.

To state our applications to the deformation of algebraic cycles, we consider for any $\mathbb{F}_p$-scheme Y its "cohomological Chow groups" and "étale-motivic cohomology groups"

$$CH^n(Y) := H^n_{\mathrm{Zar}}(Y, \mathscr{K}_{n,Y}), \qquad H^*_{\mathrm{\acute{e}t}}(Y, \mathbb{Z}_p(n)) := H^{*-n}_{\mathrm{\acute{e}t}}(Y, \{W_r\Omega^n_{Y,\log}\}_r),$$