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MONODROMY AND VINBERG FUSION FOR THE PRINCIPAL DEGENERATION OF THE SPACE OF G-BUNDLES

BY SIMON SCHIEDER

ABSTRACT. – We study the geometry and the singularities of the *principal direction* of the Drinfeld-Lafforgue-Vinberg degeneration of the moduli space of G-bundles Bun_G for an arbitrary reductive group G, and their relationship to the Langlands dual group $\check{G}$ of G.

The article consists of two parts. In the first and main part, we study the monodromy action on the nearby cycles sheaf along the principal degeneration of Bun_G and relate it to the Langlands dual group $\check{G}$. We describe the weight-monodromy filtration on the nearby cycles and generalize the results of [37] from the case $G = \mathrm{SL}_2$ to the case of an arbitrary reductive group G. Our description is given in terms of the combinatorics of the Langlands dual group $\check{G}$ and generalizations of the Picard-Lefschetz oscillators found in [37]. Our proofs in the first part use certain *local models* for the principal degeneration of Bun_G whose geometry is studied in the second part.

Our local models simultaneously provide two types of degenerations of the Zastava spaces; these degenerations are of very different nature, and together equip the Zastava spaces with the geometric analog of a Hopf algebra structure. The first degeneration corresponds to the usual Beilinson-Drinfeld fusion of divisors on the curve. The second degeneration is new and corresponds to what we call *Vinberg fusion*: it is obtained not by degenerating divisors on the curve, but by degenerating the group G via the Vinberg semigroup. Furthermore, on the level of cohomology the degeneration corresponding to the Vinberg fusion gives rise to an algebra structure, while the degeneration corresponding to the Beilinson-Drinfeld fusion gives rise to a coalgebra structure; the compatibility between the two degenerations yields the Hopf algebra axiom.

RÉSUMÉ. – Nous étudions la géométrie et les singularités de la *direction principale* de la dégénérescence de Drinfeld-Lafforgue-Vinberg de l'espace moduli de G-torseurs Bun_G pour un groupe réductif arbitraire G, et leur relation avec le groupe dual de Langlands $\check{G}$.

L'article est constitué de deux parties. Dans la première partie, nous étudions l'action de monodromie sur les cycles proches de la dégénérescence principale de Bun_G et la relions au groupe dual de Langlands $\check{G}$. Nous décrivons la filtration par monodromie sur les cycles proches et généralisons les résultats de [37] du cas $G = \mathrm{SL}_2$ au cas d'un groupe réductif arbitraire G. Notre description est donnée en termes de combinatoire du groupe dual de Langlands $\check{G}$ et de généralisations des oscillateurs de Picard-Lefschetz trouvés dans [37]. Nos preuves dans la première partie utilisent certains *modèles locaux* pour la dégénérescence principale de Bun_G dont la géométrie est étudiée dans la seconde partie.

Nos modèles locaux fournissent deux types de dégénérescence des espaces Zastava; ces dégénérations sont de nature très différente, et équipent les espaces de Zastava avec l'analogue géométrique d'une

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structure d'algèbre de Hopf. La première dégénérescence correspond à la fusion Beilinson-Drinfeld des diviseurs. La deuxième dégénérescence est nouvelle et correspond à ce que nous appelons *Vinberg fusion*: Elle est obtenue non pas par des diviseurs dégénérés sur la courbe, mais en dégénérant le groupe G via le semigroupe de Vinberg. De plus, au niveau de la cohomologie, la dégénérescence correspondant à la fusion de Vinberg donne lieu à une structure de algebra, tandis que la dégénérescence correspondant à la fusion de Beilinson-Drinfeld donne lieu à une structure de coalgebra; la compatibilité entre les deux dégénérations donne l'axiome de l'algèbre de Hopf.

1. Introduction

1.1. Context and overview

Let X be a smooth projective curve over an algebraically closed field k, let G be a reductive group over k, and let Bun_G denote the moduli stack of G-bundles on X. Drinfeld has constructed (unpublished) a canonical compactification $\overline{\mathrm{Bun}}_G$ of Bun_G which is of relevance both to the classical and geometric Langlands program; the compactification $\overline{\mathrm{Bun}}_G$ is singular, and its definition relies on the Vinberg semigroup Vin_G of G introduced by Vinberg ([39]).

While Drinfeld's definition of the compactification $\overline{\mathrm{Bun}}_G$ appeared only recently in [38], certain smooth open substacks of $\overline{\mathrm{Bun}}_G$ in the special case $G = \mathrm{GL}_n$ were already used by Drinfeld and by L. Lafforgue in their seminal work on the Langlands correspondence for function fields ([16], [15], [26]). The compactification $\overline{\mathrm{Bun}}_G$ is however already singular for $G = \mathrm{SL}_2$, and for various applications in the classical and geometric Langlands program it is necessary to understand its singularities. The study of the singularities of $\overline{\mathrm{Bun}}_G$ has begun in [37] in the case $G = \mathrm{SL}_2$, and, with a different focus, in [38] for an arbitrary reductive group G. These articles also introduce a minor modification of the space $\overline{\mathrm{Bun}}_G$ which we refer to as the *Drinfeld-Lafforgue-Vinberg degeneration* of Bun_G and denote by VinBun_G; it can be viewed as a canonical multi-parameter degeneration of Bun_G over an affine space.

The study of the singularities of $\overline{\mathrm{Bun}}_G$ and VinBun_G in [37], [38], and the present article are originally motivated by the geometric Langlands program ([24], [22]), but have also already found applications to the classical theory. As examples of applications we list the study of Drinfeld's and Gaitsgory's *miraculous duality* and *strange functional equations* in [23] and [38]; the geometric construction of the *Bernstein asymptotics map* in [38] conjectured by Sakellaridis ([35], [34], and also [6], [12]); and the geometric construction of Drinfeld's and Wang's *strange bilinear form on the space of automorphic forms* in [18] and [41], using [37] and [38], respectively. Finally, the *Picard-Lefschetz oscillators*—certain perverse sheaves found in [37] for $G = \mathrm{SL}_2$ and generalized in the present work to arbitrary reductive groups G—have recently also been shown to appear in other deformation-theoretic contexts, such as in the degeneration of Whittaker sheaves in the work of Campbell ([11]).

The work discussed in this article consists of two parts: A first and main part, and a second part which is logically independent from the first; both are concerned with the study of the geometry of the *principal degeneration* of Bun_G, a one-parameter subfamily of the multi-parameter family VinBun_G. The first part of the present work continues the study of the singularities of the space VinBun_G begun in the articles [37] and [38], though it is independent of these articles. The main theorem of the first part determines the weight-monodromy filtration on the nearby cycles sheaf of the principal degeneration of Bun_G, generalizing the main theorem of [37] from the case $G = \mathrm{SL}_2$ to the case of an arbitrary reductive group G. While this is not visible in the case $G = \mathrm{SL}_2$ treated in [37], the answer for an arbitrary reductive group achieves the passage to the Langlands dual side: Our description is given in terms of the combinatorics of the Langlands dual group $\check{G}$ of G and generalizations of the *Picard-Lefschetz oscillators* found in [37]. We refer the reader to [37, Sec. 1.3–1.5] for further background on how these results are related to the miraculous duality and the geometric Langlands program.

The proofs of the results of the first part utilize certain *local models* for the principal degeneration; the geometry of these local models is studied in further detail in a separate section. The contribution of this separate section is the construction of a novel geometric operation on the Zastava spaces that we call *Vinberg fusion* and which naturally complements the usual Beilinson-Drinfeld fusion.

1.2. The principal degeneration of Bun_G

Before discussing our main results, we first need to introduce the basic geometric objects needed for its formulation.

1.2.1. *The Vinberg semigroup* Vin_G. – In [39] Vinberg has defined and studied a canonical multi-parameter degeneration $\mathrm{Vin}_G \to \mathbb{A}^r$ of an arbitrary reductive group G of semisimple rank r, the *Vinberg semigroup*. Its fibers away from all coordinate planes are isomorphic to the group G. Its fibers over the coordinate planes afford group-theoretic descriptions in terms of the parabolic subgroups of G. While the Vinberg semigroup is singular, it possesses a certain well-behaved open subvariety which is closely related to the wonderful compactification constructed by De Concini and Procesi in [13].

1.2.2. *The Drinfeld-Lafforgue-Vinberg degeneration* VinBun_G. – As the Vinberg semigroup Vin_G comes equipped with a natural $G \times G$-action, we may form the mapping stack

$$\mathrm{Maps}(X, \mathrm{Vin}_G / G \times G)$$

parametrizing maps from the curve X to the quotient $\mathrm{Vin}_G / G \times G$. The Drinfeld-Lafforgue-Vinberg degeneration VinBun_G from [38] is then obtained from this mapping stack by imposing certain non-degeneracy conditions. The natural map $\mathrm{Vin}_G \to \mathbb{A}^r$ induces a natural map

$$\mathrm{VinBun}_G \longrightarrow \mathbb{A}^r.$$

Completely analogously to how Vin_G forms a canonical multi-parameter degeneration of the group G, this map realizes VinBun_G as a canonical multi-parameter degeneration of Bun_G. The compactification $\overline{\mathrm{Bun}}_G$ mentioned above can be obtained from VinBun_G as the quotient by a maximal torus T of G.