ISSN 0012-9593

quatrième série - tome 53 fascicule 6 novembre-décembre 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 53, 2020, p. 1393 à 1439

A NEW PROOF OF FRIEDMAN'S SECOND EIGENVALUE THEOREM AND ITS EXTENSION TO RANDOM LIFTS

BY Charles BORDENAVE

Abstract. – It was conjectured by Alon and proved by Friedman that a random d-regular graph has nearly the largest possible spectral gap, or, more precisely, that the largest absolute value of the non-trivial eigenvalues of its adjacency matrix is at most $2\sqrt{d-1}+o(1)$ with probability tending to one as the size of the graph tends to infinity. We give a new proof of this statement. We also study related questions on random n-lifts of graphs and improve a recent result by Friedman and Kohler.

Résumé. – Il a été conjecturé par Alon et démontré par Friedman qu'un graphe d-régulier aléatoire a un trou de spectre asymptotiquement maximal, ou, plus précisément, que la plus grande valeur propre non-triviale de sa matrice d'adjacence est au plus $2\sqrt{d-1}+o(1)$ avec probabilité tendant vers un lorsque la taille du graphe tend vers l'infini. Nous donnons une nouvelle preuve de ce résultat. Nous étudions aussi des questions reliées sur les n-revêtements aléatoires d'un graphe et améliorons un résultat récent de Friedman and Kohler.

1. Introduction

Let us consider a finite simple graph $G = (V, E)$ with $n = |V|$ vertices. Its adjacency matrix $A = A(G)$ is the matrix indexed by V and defined for all $u, v \in V$ by $A_{uv} = \mathbb{1}_{\{u,v\}\in E}$ where $\mathbb{1}$ denotes the indicator function. The matrix A is symmetric, its eigenvalues $\mu_i = \mu_i(G)$ are real and we order them non-increasingly,

$$\mu_n \leq \cdots \leq \mu_1.$$

We assume further that, for some integer $d \geq 3$, the graph G is d-regular, that is, all vertices have degree d. We then have that $\mu_1 = d$, that all eigenvalues have absolute value at most d, and $\mu_n = -d$ is equivalent to G having a bipartite connected component. The absolute value of the largest non-trivial eigenvalues of G is denoted by $\mu = \mu(G) = \max\{|\mu_i| : |\mu_i| < d\}$. Classical statements such as Cheeger's isoperimetric inequality or Chung's diameter inequality relate small values of μ or μ_2 with good expanding properties of the graph G, we

The author is supported by the research grants ANR-14-CE25-0014 and ANR-16-CE40-0024-01.

doi:10.24033/asens.2450

refer for example to [10, 15]. It turns out that μ cannot be made arbitrarily small. Indeed, a celebrated result of Alon-Boppana implies that for any d-regular graph with n vertices,

$$\mu_2(G) \geq 2\sqrt{d-1} - \varepsilon_d(n), \tag{1}$$

where, for some constant $c_d > 0$, $\varepsilon_d(n) = c_d/(\log n)^2$; see the above references and [28, 12, 26]. Following [22, 20], one may try to construct graphs which achieve the Alon-Boppana bound. A graph is called Ramanujan if $\mu \leq 2\sqrt{d-1}$. Proving the existence of Ramanujan graphs with a large number of vertices is a difficult task which has been solved for arbitrary $d \geq 3$ only recently [21]. On the other end, it was conjectured by Alon [2] and proved by Friedman [12] that most d-regular graphs are weakly Ramanujan. More precisely, for integer $n \geq 1$, we define $\mathcal{G}_d(n)$ as the set of simple d-regular graphs with vertex set $\{1, \dots, n\}$. If nd is even and $d \leq n-1$, this set is non-empty (for nd odd, a definition of $\mathcal{G}_d(n)$ is given in [12]). A uniformly sampled d-regular graph is then a random graph whose distribution is uniform on $\mathcal{G}_d(n)$.

THEOREM 1 (Friedman's second eigenvalue theorem [12]). – *Let $d \geq 3$ be an integer and nd be even. If G is uniformly distributed on $\mathcal{G}_d(n)$, we have for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\mu_2 \vee |\mu_n| \geq 2\sqrt{d-1} + \varepsilon\Big) = 0,$$

where $a \vee b = \max(a, b)$ and the limit is along any sequence going to infinity with nd even.

The first aim of this paper is to give a new proof of this result. The argument detailed in Section 2 simplifies substantially the original proof. A careful reading of the proof actually gives the following quantitative statement: for any $0 < a < 1$, there exists $c > 0$ (depending on d and a) such that for all integers n such that $\mathcal{G}_d(n)$ is non-empty,

$$\mathbb{P}\left(\mu_2 \vee |\mu_n| \geq 2\sqrt{d-1} + c\left(\frac{\log\log n}{\log n}\right)^2\right) \leq n^{-a}. \tag{2}$$

The method is robust and it has been recently applied in [7] to random graphs with structure (stochastic block model).

The second aim of this paper is to apply this method to study similar questions on the eigenvalues of random lifts of graphs. This class of models sheds a new light on Ramanujan-type properties, and, since the work of Amit and Linial [3, 4] and Friedman [11], it has attracted a substantial attention [17, 1, 19, 30, 13]. To avoid any confusion in notation, we will postpone to Section 3 the precise definition of random lifts and the statement of the main results. In Section 4 we will give a simpler proof of a recent result of Friedman and Kohler [13] and establish a weak Ramanujan property for the non-backtracking eigenvalues of a random lift of an arbitrary graph.

Notation. – If n is a positive integer, we set $[n] = \{1, \dots, n\}$. If $M \in M_n(\mathbb{R})$, M^* denotes its conjugate transpose and we denote its operator norm by

$$\|M\| = \sup_{x\in\mathbb{R}^n, x\neq 0} \frac{\|Mx\|_2}{\|x\|_2}.$$

For positive sequences a_n, b_n, we will use the standard notation

$$\begin{aligned} a_n &\sim b_n && \text{if } \lim_{n\to\infty} a_n/b_n = 1, \\ a_n &= O(b_n) && \text{if } \limsup_{n\to\infty} a_n/b_n < \infty \\ \text{and } a_n &= o(b_n) && \text{if } \lim_{n\to\infty} a_n/b_n = 0. \end{aligned}$$

Finally, we shall write that an event Ω_n holds *with high probability*, w.h.p. for short, if $\mathbb{P}(\Omega_n^c) = o(1)$.

2. Proof of Theorem 1

2.1. Overview of the proof

Let us describe the strategy of proof of Theorem 1 and its main difficulties. Following Füredi and Komlós [14] and Broder and Shamir [8], a natural strategy is to estimate the trace of a high power of the adjacency matrix. Namely, if we manage to prove that w.h.p.

$$\text{tr}(A^k) \le d^k + n\left(2\sqrt{d-1} + o(1)\right)^k \tag{3}$$

for some even integer $k = k(n)$ such that $k \gg \log n$ then Theorem 1 would follow. Indeed, from the spectral theorem, (3) implies that w.h.p.

$$\mu_2^k + \mu_n^k \le \text{tr}(A^k) - d^k \le n\left(2\sqrt{d-1} + o(1)\right)^k.$$

Therefore, w.h.p.

$$\mu_2 \vee |\mu_n| \le n^{1/k}\left(2\sqrt{d-1} + o(1)\right) = 2\sqrt{d-1} + o(1),$$

where the last equality comes from $n^{1/k} = 1 + o(1)$. From Serre [31], we note that for any $\varepsilon > 0$, there is a positive proportion of the eigenvalues of A which are larger than $2\sqrt{d-1} - \varepsilon$. This explains the necessary presence of the factor n on the right-hand side of (3). Observe also that the entries of the matrix A^k count the number of paths of length k between two vertices. Since $k \gg \log n$, we are interested in the asymptotic number of closed paths of length k when k is much larger than the typical diameter of the graph.

To avoid the presence of d^k on the right-hand side of (3), we may project A onto the orthogonal complement of the eigenspace associated to $\mu_1 = d$ and then compute the trace. If J is the $n \times n$ matrix with all entries equal to 1, we should then prove that w.h.p. for some even k, $k \gg \log n$,

$$\text{tr}(A^k) - d^k = \text{tr}\left(A - \frac{d}{n}J\right)^k \le n\left(2\sqrt{d-1} + o(1)\right)^k. \tag{4}$$

The main difficulty hidden behind Friedman's Theorem 1 is that statements (3)-(4) do not hold in expectation for $k \gg \log n$. This is due to the presence of subgraphs in the graph which occur with polynomially small probability. For example, it follows from McKay [24] that for n large enough, the graph contains as subgraph the complete graph with $d+1$ vertices