ISSN 0012-9593

quatrième série - tome 53 fascicule 6 novembre-décembre 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Tristan BOZEC & Olivier SCHIFFMANN & Éric VASSEROT

On the number of points of nilpotent quiver varieties over finite fields

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 428 euros.
Abonnement avec supplément papier :
Europe : 576 €. Hors Europe : 657 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 53, 2020, p. 1501 à 1544

ON THE NUMBER OF POINTS OF NILPOTENT QUIVER VARIETIES OVER FINITE FIELDS

BY Tristan BOZEC, Olivier SCHIFFMANN AND Éric VASSEROT

ABSTRACT. – We give a closed expression for the number of points over finite fields of the Lusztig nilpotent variety associated to any quiver, in terms of Kac's A-polynomials. When the quiver has 1-loops or oriented cycles, there are several possible variants of the Lusztig nilpotent variety, and we provide formulas for the point count of each. This involves nilpotent versions of the Kac A-polynomial, which we introduce and for which we give a closed formula similar to Hua's formula for the usual Kac A-polynomial. Finally we compute the number of points over a finite field of the various strata of the Lusztig nilpotent variety involved in the geometric realization of the crystal graph.

RÉSUMÉ. – Nous donnons une formule close exprimant le nombre de $\mathbb{F}_q$-points des variétés nilpotentes de Lusztig associées à un carquois quelconque en termes des A-polynômes de Kac.

Lorsque le carquois possède des 1-cycles ou des cycles orientés, il existe plusieurs variantes des variétés nilpotentes de Lusztig ; nous fournissons des formules pour le nombre de $\mathbb{F}_q$ points dans tous les cas. Ceci fait intervenir des variantes nilpotentes des polynômes de Kac que nous définissons et pour lesquels nous donnons une formule similaire à la formule de Hua pour les polynômes de Kac usuels. Enfin, nous calculons également les nombres de $\mathbb{F}_q$-points des diverses strates de la variété nilpotente de Lusztig impliquées dans la réalisation géométrique des graphes de cristaux. Nous en déduisons une démonstration de l'analogue, pour un carquois arbitraire, de la conjecture de Kac liant caractère des algèbres de Kac-Moody et terme constant des A-polynomes.

0. Introduction

The interplay between the geometry of moduli spaces of representations of quivers and the representation theory of quantum groups has led to numerous constructions and results of fundamental importance for both areas. One of the central objects in the theory is the *Lusztig nilpotent variety* introduced in [22], which is a closed substack $\underline{\Lambda}_{\mathbf{d}}$ of the cotangent stack $T^*\underline{\mathrm{Rep}}_{\mathbf{d}}(Q)$ of the stack of representations of dimension $\mathbf{d}$ of a quiver Q. When Q has no 1-cycle the stack $\underline{\Lambda}_{\mathbf{d}}$ is Lagrangian and, as shown by Lusztig (resp. Kashiwara-Saito), its irreducible components are in one to one bijection with the weight $\mathbf{d}$ piece of the canonical

doi:10.24033/asens.2452

basis (resp. crystal graph) of $U_q^+(\mathfrak{g}_Q)$, where $\mathfrak{g}_Q$ is the Kac-Moody Lie algebra associated to Q. The stack $\underline{\Lambda}_{\mathbf{d}}$ is singular, and although it can be inductively built by sequences of (stratified) affine fibrations, see [19], its geometry remains mysterious. The link mentioned above with canonical or crystal bases shows that, for quivers without 1-cycles, the generating series for *top* Borel-Moore homology groups of $\underline{\Lambda}_{\mathbf{d}}$ is given by

$$(0.1)\qquad \sum_{\mathbf{d}} \dim(H_{top}(\underline{\Lambda}_{\mathbf{d}}, \mathbb{Q}))\, z^{\mathbf{d}} = \sum_{\mathbf{d}} \dim(U^+(\mathfrak{g}_Q)[\mathbf{d}])\, z^{\mathbf{d}} = \prod_{\alpha \in \Delta^+} (1 - z^{\alpha})^{-\dim \mathfrak{g}_Q[\alpha]}.$$

A natural problem is to extend the formula (0.1) to the whole cohomology of $\underline{\Lambda}_{\mathbf{d}}$ and to understand its significance from the point of view of representation theory. One first step in this program is worked out in this paper together with its companion [35]. The aim of the present paper is to carry out the essential step in the computation of the cohomology of $\underline{\Lambda}_{\mathbf{d}}$: we determine the number of points of $\underline{\Lambda}_{\mathbf{d}}$ over finite fields of large enough characteristic. The answer, given in the form of generating series, is expressed in terms of the Kac polynomials $A_{\mathbf{d}}$ attached to the quiver Q. We refer the reader to Theorem 1.4 for details. In [35] it is proven that $\underline{\Lambda}_{\mathbf{d}}$ is cohomologically pure, and hence that its Poincaré polynomial coincides with its counting polynomial. In addition, the (whole) cohomology of $\underline{\Lambda}_{\mathbf{d}}$ is related there to the Lie algebras introduced by Maulik and Okounkov in [24].

Our strategy to compute the number of points of $\underline{\Lambda}_{\mathbf{d}}(\mathbb{F}_q)$ is the following : we relate the number of points of $\underline{\Lambda}_{\mathbf{d}}(\mathbb{F}_q)$ to the number of points of certain *Lagrangian* Nakajima quiver varieties $\mathfrak{L}(\mathbf{d}, \mathbf{n})(\mathbb{F}_q)$. Using some purity result for these Nakajima quiver varieties together with a Poincaré duality argument we express the counting polynomial of $\mathfrak{L}(\mathbf{d}, \mathbf{n})$ in terms of the Poincaré polynomial of the *symplectic* Nakajima quiver variety $\mathfrak{M}(\mathbf{d}, \mathbf{n})$. Finally, we use Hausel's computation of the Poincaré polynomials of $\mathfrak{M}(\mathbf{d}, \mathbf{n})$ in terms of Kac polynomials, see [12].

We note that the generating series for the top homology groups of $\underline{\Lambda}_{\mathbf{d}}$ can be extracted from our formula, and involves only the constant terms of Kac polynomials : combining this with (0.1) one recovers a proof of Kac's conjecture (first proved in [12]) relating the multiplicities of root spaces in Kac-Moody algebras to the constant term of Kac polynomials.

In the context of [24] it is essential to allow for *arbitrary* quivers Q, such as for instance the quiver with one vertex and g loops. Note that there is no Kac-Moody algebra associated to a quiver which does carry 1-cycles. In [1], [2] the first author introduced a quantum group $U_q(\mathfrak{g}_Q)$ attached to an arbitrary quiver Q which coincides with the usual quantized Kac-Moody algebra for a quiver with no 1-cycles and, he generalized to this context several fundamental constructions and results, in particular the theory of canonical and crystal bases, and an analogue of (0.1). In the presence of 1-cycles, Lusztig's nilpotent variety is *not* Lagrangian anymore and one has to consider instead a larger subvariety Λ^1 defined by some *'semi-nilpotency'* condition. In addition, when the quiver Q contains some oriented cycle we consider yet a third subvariety Λ^0 defined by some weak form of semi-nilpotency. The varieties Λ^0, Λ^1 are Lagrangian, contain Λ and are in some sense more natural than Λ from a geometric perspective. We carry out in parallel the computation of the number of points over finite fields for each of the Λ^0, Λ^1 and Λ. This leads us to introduce two variants A^0 and A^1 of the Kac polynomials, respectively counting nilpotent and 1-cycle nilpotent indecomposable representations, see Section 1.4 for more details, for which we prove the existence and give an explicit formula similar to Hua's formula. As an application, we provide a proof of an extension of Kac's conjecture on the multiplicities of Kac-Moody Lie algebras to the setting of arbitrary quivers.

To finish, let us briefly describe the contents of this paper : the main actors are introduced and our main theorem is stated in Section 1, where several examples are explicitly worked out. Section 2 deals with the existence of nilpotent Kac polynomials A^0, A^1, and provides explicit formulas for these in the spirit of Hua's formula for the usual Kac polynomial. In Section 3 we study several subvarieties $\mathfrak{L}^0(\mathbf{v},\mathbf{w})$, $\mathfrak{L}(\mathbf{v},\mathbf{w})$, $\mathfrak{L}^1(\mathbf{v},\mathbf{w})$ of the symplectic Nakajima quiver variety $\mathfrak{M}(\mathbf{v},\mathbf{w})$, respectively corresponding to the Lusztig nilpotent varieties Λ^0, Λ^1 and Λ. More precisely, we establish some purity results and compute the counting polynomials of these subvarieties by combining a Poincaré duality argument (based on Byalinicki-Birula decompositions) with Hausel's computation of the Betti numbers of $\mathfrak{M}(\mathbf{v},\mathbf{w})$. In Section 4 we relate the counting polynomials of $\Lambda_{\mathbf{v}}$, $\Lambda^0_{\mathbf{v}}$, $\Lambda^1_{\mathbf{v}}$ to the counting polynomials of $\mathfrak{L}(\mathbf{v},\mathbf{w})$, $\mathfrak{L}^0(\mathbf{v},\mathbf{w})$, $\mathfrak{L}^1(\mathbf{v},\mathbf{w})$ and prove our main theorem. Section 5 contains an observation about the counting polynomials of certain natural strata in Lusztig nilpotent varieties arising in the geometric realization of crystal graphs. Finally, in the appendix we recall the definition of the quantum group associated in [2] to an arbitrary quiver, give a character formula for it, and use our main theorem to prove an extension of Kac's conjecture in that context.

1. Statement of the result

1.1. Lusztig nilpotent quiver varieties

Let $Q=(I,\Omega)$ be a finite [(1)] quiver, with vertex set I and edge set H. For $h\in\Omega$ we will denote by h',h'' the initial and terminal vertex of h. Note that we allow 1-loops, i.e., edges h satisfying $h'=h''$. Set $\mathbf{v}\cdot\mathbf{v}'=\sum_i v_i v_i'$. We denote by

$$\langle\mathbf{v},\mathbf{v}'\rangle=\mathbf{v}\cdot\mathbf{v}'-\sum_{h\in\Omega}v_{h'}v'_{h''}$$

the Euler form on $\mathbb{Z}^I$, and by $(\bullet,\bullet)$ its symmetrized version such that $(\mathbf{v},\mathbf{v}')=\langle\mathbf{v},\mathbf{v}'\rangle+\langle\mathbf{v}',\mathbf{v}\rangle$. We will call *imaginary* (resp. *real*) a vertex which carries a 1-loop (resp. which doesn't carry an 1-loop) and write $I=I^{\mathrm{im}}\sqcup I^{\mathrm{re}}$ for the associated partition of I.

Let $Q^*=(I,\Omega^*)$ be the opposite quiver, in which the direction of every arrow is inverted. Let $\bar{Q}=(I,\bar{\Omega})$ with $\bar{\Omega}=\Omega\sqcup\Omega^*$ be the doubled quiver, obtained from Q by replacing each arrow h by a pair of arrows (h,h^*) going in opposite directions.

Fix a field k. For each dimension vector $\mathbf{v}\in\mathbb{N}^I$ we fix an I-graded k-vector space $V=\bigoplus_i V_i$ of graded dimension $\mathbf{v}$ and we set

$$E_{\mathbf{v}}=\bigoplus_{h\in\Omega}\mathrm{Hom}(V_{h'},V_{h''}),\quad E^*_{\mathbf{v}}=\bigoplus_{h\in\Omega^*}\mathrm{Hom}(V_{h'},V_{h''}),\quad \bar{E}_{\mathbf{v}}=\bigoplus_{h\in\bar{\Omega}}\mathrm{Hom}(V_{h'},V_{h''}).$$

Elements of $E_{\mathbf{v}}$, $E^*_{\mathbf{v}}$ and $\bar{E}_{\mathbf{v}}$ will be denoted by $x=(x_h)$, $x^*=(x_{h^*})$ and $\bar{x}=(x,x^*)$.

By a flag of I-graded vector spaces in V we mean a finite increasing flag of I-graded subspaces $\big(\{0\}=L^0\subsetneq L^1\subsetneq\cdots\subsetneq L^s=V\big)$. We'll say that (L^l) is a *restricted* flag of I-graded vector spaces if for all l the vector space L^l/L^{l-1} is concentrated on one vertex.

(1) Locally finite quiver would do as well.