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K-STABILITY OF FANO SPHERICAL VARIETIES

BY THIBAUT DELCROIX

ABSTRACT. – We prove a combinatorial criterion for K-stability of a $\mathbb{Q}$-Fano spherical variety with respect to equivariant special test configurations, in terms of its moment polytope and some combinatorial data associated to the open orbit. Combined with the equivariant version of the Yau-Tian-Donaldson conjecture for Fano manifolds proved by Datar and Székelyhidi, it yields a criterion for the existence of a Kähler-Einstein metric on a spherical Fano manifold. The results hold also for modified K-stability and existence of Kähler-Ricci solitons.

RÉSUMÉ. – Nous prouvons, pour une variété sphérique $\mathbb{Q}$-Fano, un critère combinatoire de K-stabilité par rapport aux configurations test spéciales équivariantes, exprimé en fonction de son polytope moment et d'une donnée combinatoire associée à l'orbite ouverte. En utilisant la version équivariante de la conjecture de Yau-Tian-Donaldson prouvée par Datar et Székelyhidi, cela devient un critère d'existence de métriques Kähler-Einstein sur les variétés sphériques Fano lisses. Les résultats s'appliquent également à la K-stabilité modifiée et à l'existence de solitons de Kähler-Ricci.

Introduction

Kähler-Einstein metrics on a Kähler manifold are the solutions (if they exist) of a highly non linear second order partial differential equation on the manifold. It is not clear at the moment under which conditions the equation admits solutions on a Fano manifold. In the recent years a major advance in this direction has been made through the resolution of the Yau-Tian-Donaldson conjecture in the Fano case, by Chen, Donaldson, and Sun [18, 19, 20]. This conjecture states in its general form that the existence of some canonical metrics on a Kähler manifold should be related to the algebro-geometric condition of K-stability on the manifold.

The K-stability condition is a condition involving the positivity of numerical invariants associated to polarized one parameter degenerations of the manifold, equipped with an action of $\mathbb{C}^*$, called *test configurations*. In the Fano case, it was proved by Li and Xu [40] (and also Chen, Donaldson and Sun) that it is enough to consider test configurations with

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normal central fiber, which are called *special* test configurations. Other proofs of the Yau-Tian-Donaldson conjecture were obtained by Tian [57], Datar and Székelyhidi [23], Chen, Sun and Wang [21], Berman, Boucksom and Jonsson [8]. The work of Datar and Székelyhidi is of special interest to us as it allows us to take into account automorphisms of the manifold, by considering only equivariant test configurations, and extends the result to Kähler-Ricci solitons.

The necessity of the K-stability condition with respect to special test configurations was established earlier by Tian who provided an example of Fano manifold with vanishing Futaki invariant but no Kähler-Einstein metrics [56]. It was not clear at first if the K-stability condition could be used to prove the existence of Kähler-Einstein metrics on explicit examples of Fano manifolds. One aim of the present article is to provide an illustration of the power of the approach to the existence of Kähler-Einstein metrics *via* K-stability, on highly symmetric manifolds.

Namely, we obtain a criterion for the existence of Kähler-Einstein metrics on a Fano spherical manifold, involving only the moment polytope and the valuation cone of the spherical manifold. Both are classical and central objects in the theory of spherical varieties. The class of spherical varieties is a very large class of highly symmetric varieties, which contains toric varieties, generalized flag manifolds, homogeneous toric bundles, biequivariant compactifications of reductive groups. For all of these subclasses for which a criterion was known, our result specializes to the same criterion (compare with [62, 51, 25]). Examples of new varieties to which our criterion applies include colored horospherical varieties and symmetric varieties (for examples the varieties constructed in [24]). Let us mention here the work of Ilten and Süss on Fano manifolds with an action of a torus with complexity one [35], and also the discussion in [23, Section 4], which were to the author's knowledge the first applications of K-stability as a sufficient criterion for explicit examples.

Another aim of this article is to provide a framework for understanding the K-stability condition in this class of spherical varieties, which should lead to a better understanding of K-stability in general. The author obtained in [25, 26] an example of group compactification with no Kähler-Einstein metric but vanishing Futaki invariant, which is furthermore not K-semistable unlike the Mukai-Umemura type example from [56]. This is evidence that non trivial K-stability phenomena appear in the class of spherical Fano manifolds, which was not true for toric Fano manifolds.

Before stating the main result of the article, let us introduce some notations, the moment polytope and the valuation cone.

Let G be a complex connected reductive algebraic group. Let B be a Borel subgroup of G and T a maximal torus of B. Let $\mathfrak{X}(T)$ denote the group of algebraic characters of T. Denote by $\Phi \subset \mathfrak{X}(T)$ the root system of (G, T) and Φ^+ the positive roots determined by B.

Let X be a Fano manifold, spherical under the action of G, which means that B acts on X with an open and dense orbit. The *moment polytope* $\Delta^+ \subset \mathfrak{X}(T) \otimes \mathbb{R}$ of X with respect to B is a polytope encoding the structure of representation of G on the spaces of sections of tensor powers of the anticanonical line bundle. Alternatively, from a symplectic point of view, it can be characterized as the Kirwan moment polytope of (X, ω) with respect to the action of a maximal compact subgroup K of G, where ω is a K-invariant Kähler form in $c_1(X)$ (see [12]). The moment polytope Δ^+ determines a sub-root system Φ_L of Φ, composed of those roots that are orthogonal to the affine span of Δ^+ with respect to the *Killing form* κ. Let Φ_{Q^u} be the set $\Phi^+ \setminus \Phi_L$, and $2\rho_Q$ be the sum of the elements of Φ_{Q^u}.

A spherical variety X also has an open and dense orbit O under the action of G. The valuation cone of X depends only on this open orbit O. Let $\mathcal{M} \subset \mathfrak{X}(T)$ be the set of characters of B-semi-invariant functions in the function field $\mathbb{C}(O)$ of O, and let $\mathcal{N}$ be its $\mathbb{Z}$-dual. The restriction of a $\mathbb{Q}$-valued valuation on $\mathbb{C}(O)$ to the B-semi-invariant functions defines an element of $\mathcal{N} \otimes \mathbb{Q}$. The *valuation cone* $\mathcal{V}$ with respect to B is defined as the set of those elements of $\mathcal{N} \otimes \mathbb{Q}$ induced by G-invariant valuations on $\mathbb{C}(O)$.

Remark that the vector space $\mathcal{N} \otimes \mathbb{Q}$ is a quotient of the vector space $\mathfrak{Y}(T) \otimes \mathbb{Q}$, where $\mathfrak{Y}(T)$ is the group of algebraic one parameter subgroups of T. Denote by $\pi : \mathfrak{Y}(T) \otimes \mathbb{Q} \longrightarrow \mathcal{N} \otimes \mathbb{Q}$ the quotient map, so that $\pi^{-1}(\mathcal{V}) \subset \mathfrak{Y}(T) \otimes \mathbb{Q}$. Let $\Xi \subset \mathfrak{X}(T) \otimes \mathbb{R}$ be the dual cone to the closure of the inverse image by π of the opposite of the valuation cone $\pi^{-1}(-\mathcal{V})$ in $\mathfrak{Y}(T) \otimes \mathbb{R}$ (dual with respect to the extension of the natural pairing $\langle \cdot, \cdot \rangle : \mathfrak{X}(T) \times \mathfrak{Y}(T) \longrightarrow \mathbb{Z}$).

The group of G-equivariant automorphisms $\mathrm{Aut}_G(X)$ of the spherical manifold X is diagonalizable. The real vector space generated by the linear part of $\mathcal{V}$ is in fact isomorphic to $\mathfrak{Y}(\mathrm{Aut}_G(X)) \otimes \mathbb{R}$.

Given ζ in $\mathfrak{Y}(\mathrm{Aut}_G(X)) \otimes \mathbb{R}$ identified with an element of $\mathcal{N} \otimes \mathbb{R}$, and a choice $\tilde{\xi} \in \pi^{-1}(\zeta)$ of lift of ζ, we denote by $\mathrm{bar}_{DH,\tilde{\xi}}(\Delta^+)$ the barycenter of the polytope Δ^+ with respect to the measure with density $p \mapsto e^{2\langle p-2\rho_Q, \tilde{\xi}\rangle} \prod_{\alpha \in \Phi_{Q^u}} \kappa(\alpha, p)$ with respect to the Lebesgue measure dp on $\mathfrak{X}(T) \otimes \mathbb{R}$. Our main result is the following.

THEOREM A. – *Let X be a Fano spherical manifold. The following are equivalent.* 1.1.

1. There exists a Kähler-Ricci soliton on X with associated holomorphic vector field ζ.
2. The barycenter $\mathrm{bar}_{DH,\tilde{\xi}}(\Delta^+)$ is in the relative interior of the cone $2\rho_Q + \Xi$.
3. The manifold X is modified K-stable with respect to equivariant special test configurations.
4. The manifold X is modified K-stable.

The equivalence between (1) and (4) holds for any Fano manifold. In the setting of Kähler-Einstein metrics, it is the consequence on one hand of the work of Chen, Donaldson and Sun recalled earlier, and on the other hand of the work of Berman [7]. In the more general setting of Kähler-Ricci solitons, Berman and Witt-Nystrom [9] showed that (4) is a necessary condition for (1). Datar and Székelyhidi [23] showed that (3) implies (1) for any Fano manifold equipped with an action of a complex reductive group, and (4) clearly implies (3). What we prove in this article is the equivalence between (2) and (3) in the case of a spherical Fano manifold. Furthermore we prove that the equivalence between (2) and (3) holds for singular $\mathbb{Q}$-Fano spherical varieties.

The intuition for our main result came from our previous work on group compactifications, which did not involve K-stability. The proof of a Kähler-Einstein criterion for smooth and Fano group compactifications in [25, 26] can be adapted to provide another proof of the criterion for Kähler-Ricci solitons on the same manifolds. Similarly, Wang-Zhu type methods (as used in [62] and [51]), together with some results proved for horospherical