ISSN 0012-9593

quatrième série - tome 53 fascicule 3 mai-juin 2020

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Valentino TOSATTI & Yuguang ZHANG

Collapsing Hyperkähler manifolds

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 53, 2020, p. 751 à 786

COLLAPSING HYPERKÄHLER MANIFOLDS

BY Valentino TOSATTI AND Yuguang ZHANG

ABSTRACT. – Given a projective hyperkähler manifold with a holomorphic Lagrangian fibration, we prove that hyperkähler metrics with volume of the torus fibers shrinking to zero collapse in the Gromov-Hausdorff sense (and smoothly away from the singular fibers) to a compact metric space which is a half-dimensional special Kähler manifold outside a singular set of real Hausdorff codimension 2, and is homeomorphic to the base projective space.

RÉSUMÉ. – Étant donné une variété hyperkählérienne projective avec une fibration lagrangienne holomorphe, on montre que les métriques hyperkählériennes avec le volume des fibres s'approchant de zéro, s'effondrent au sens de Gromov-Hausdorff (et de façon lisse en dehors des fibres singulières) sur un espace métrique compact qui est une variété kählérienne spéciale demi-dimensionnelle en dehors d'un ensemble singulier de codimension réelle de Hausdorff 2, et est homéomorphe à un espace projectif.

1. Introduction

Let M^m be a compact Calabi-Yau manifold, which for us is a compact Kähler manifold M^m with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Yau's Theorem [66] shows that given any Kähler class $[\alpha]$ on M we can find a unique representative ω of $[\alpha]$ which is a *Ricci-flat Kähler metric*. The basic problem that we study in this paper is to understand the limiting behavior of such Ricci-flat metrics if we degenerate the class $[\alpha]$. More precisely, we fix a class $[\alpha_0]$ on the boundary of the Kähler cone and for $0 < t \leqslant 1$ we let $\tilde{\omega}_t$ be the unique Ricci-flat Kähler metric in the class $[\alpha_0] + t[\omega_M]$, where ω_M is a fixed Ricci-flat Kähler metric on M, and we wish to understand the behavior of $(M, \tilde{\omega}_t)$ as $t \to 0$. The metrics $\tilde{\omega}_t$ satisfy the equation

$$\tilde{\omega}_t^m = c_t t^{m-n} \omega_M^m, \tag{1.1}$$

for some explicit constants c_t which approach a positive constant as $t \to 0$. Up to scaling the whole setup, we may assume without loss of generality that $c_t \to 1$ when $t \to 0$.

 doi:10.24033/asens.2433

This question has been extensively studied in the literature, the most relevant works being [25, 58, 59, 23, 24, 27, 63, 65, 9, 54], see also the surveys [60, 61, 69]. In particular, decisive results in the *non-collapsing* case when $\int_M \alpha_0^n > 0$ have been obtained in [58, 9, 54]. In this paper we consider the more challenging *collapsing* case when $\int_M \alpha_0^n = 0$, and we will always assume that $[\alpha_0] = f^*[\omega_N]$ where (N^n, ω_N) is a compact Kähler manifold with $0 < n < m$ and $f : M \to N$ is a holomorphic surjective map with connected fibers (i.e., a fiber space). This is the same setup as in [59, 23, 24, 27, 63, 65], and as explained there, in this case there are proper analytic subvarieties $S' \subset N$ and $S = f^{-1}(S') \subset M$ such that $f : M\backslash S \to N_0 := N\backslash S'$ is a proper submersion with fibers $M_y = f^{-1}(y), y \in N\backslash S'$, smooth Calabi-Yau $(m-n)$-folds.

In [63], building upon the earlier [59], it is shown that there is a Kähler metric ω on N_0 such that as $t \to 0$ the metrics $\tilde{\omega}_t$ converges to $f^*\omega$ uniformly on compact subsets of $M\backslash S$, and ω satisfies

$$\mathrm{Ric}(\omega) = \omega_{WP} \geqslant 0, \tag{1.2}$$

on N_0, where ω_{WP} is a *Weil-Petersson form* which measures the variation of the complex structures of the fibers M_y (see e.g., [59, 55]). This is improved to smooth convergence on compact subsets of $M\backslash S$ in [23, 27, 65] when the fibers M_y are tori (or finite étale quotients of tori). Explicit estimates are also obtained for $\tilde{\omega}_t$ near S, but these blow up very fast near S.

Our main concern is understanding the possible collapsed Gromov-Hausdorff limits of $(M, \tilde{\omega}_t)$ as $t \to 0$, and their singularities. In this regard, we have the following conjecture (see [60, Question 4.4] [61, Question 6]), which is motivated by an analogous conjecture by Gross-Wilson [25], Kontsevich-Soibelman [37, 38] and Todorov [42] for collapsed limits of Ricci-flat Kähler metrics on Calabi-Yau manifolds near a large complex structure limit:

Conjecture 1.1. – *If (X, d_X) denotes the metric completion of (N_0, ω), and $S_X = X\backslash N_0$, then*

(a) *(X, d_X) is a compact length metric space and S_X has real Hausdorff codimension at least* 2.

(b) *We have that*

$$(M, \tilde{\omega}_t) \xrightarrow{d_{GH}} (X, d_X),$$

when $t \to 0$.

(c) *X is homeomorphic to N.*

This conjecture was proved by Gross-Wilson [25] when $f : M \to N$ is an elliptic fibration of $K3$ surfaces with only I_1 singular fibers. In our earlier work with Gross [24] we proved Conjecture 1.1 completely in the case when $\dim N = 1$. Very recently, conditionally to a certain Hölder estimate for solutions of a family of Monge-Ampère equations, a new proof was obtained in [40] when $\dim M = 3, \dim N = 1$, the generic fibers M_y are $K3$ surfaces and the singular fibers are nodal $K3$ surfaces, which also gives better estimates near and on the singular fibers.

For bases N of general dimension n, the only known partial result towards Conjecture 1.1 is the one proved in [63, 23]: if (X, d_X) is the Gromov-Hausdorff limit of a sequence $(M, \tilde{\omega}_{t_i}), t_i \to 0$ (such limits always exist up to passing to subsequences), then there is a homeomorphism $\psi : N_0 \to X_0$ onto a dense open subset $X_0 \subset X$ such that $\psi : (N_0, \omega) \to (X_0, d_X|_{X_0})$ is a local isometry.

Our main result is the following:

THEOREM 1.2. – *Conjecture 1.1 holds when M is a projective hyperkähler manifold.*

As proved in [24], in this case the limiting metric ω on N_0 is a *special Kähler metric* in the sense of [13]. In this case, the base N is always $\mathbb{CP}^n$ [31] and the fibers M_y, $y \in N_0$, are holomorphic Lagrangian n-tori [45, 46], so that f is an *algebraic completely integrable system* over N_0. A classical result of Donagi-Witten [12] (see also [13, 30]) shows that the base of an algebraic completely integrable system admits a special Kähler metric, and our result shows that this metric arises as the collapsed limit of hyperkähler metrics on the total space.

An application of our result is the revised Strominger-Yau-Zaslow (SYZ) conjecture due to Gross-Wilson [25], Kontsevich-Soibelman [37, 38] and Todorov [42] (note that the statement of the conjecture in [37, 38] also covers the hyperkähler case). As explained in [23], Theorem 1.2 implies a positive solution to such conjecture for collapsed limits of hyperkähler metrics near large complex structure limits which arise via hyperkähler rotation from our setting above:

COROLLARY 1.3. – *The conjecture of Gross-Wilson* [25, Conjecture 6.2]*, Kontsevich-Soibelman* [37, Conjectures 1 and 2] *and Todorov* [42, p. 66] *holds for those large complex structure families of hyperkähler manifolds which arise from the setup of Theorem 1.2 via hyperkähler rotation as in* [23, Theorem 1.3].

Indeed, this follows exactly as in [23, Theorem 1.3], using Theorem 1.2 together with our earlier results in [24, Theorem 1.2]. The key new information provided by Theorem 1.2, which was not available in [23, 24], is the uniqueness of the Gromov-Hausdorff limit, which is identified with the metric completion of the smooth part and is homeomorphic to the base, and the fact that it has singularities in real codimension at least 2. This completes the program we started in [23] to extend Gross-Wilson's theorem on large complex structure limits of $K3$ surfaces [25] to higher-dimensional hyperkähler manifolds. Furthermore, by combining it with [24], Theorem 1.2 gives more precise information about the limit space, which was predicted by [25, 37, 38]. This is explained in detail in Section 5 (see Theorem 5.2 there) in a slightly more general setup than [23].

We now give a brief outline of the paper. In Section 2 we extend and sharpen a method introduced in our earlier work [24] (when $\dim N = 1$) and show that to prove parts (a) and (b) of Conjecture 1.1 in general it suffices to obtain an upper bound for the limiting metric ω near S' (which may be assumed to be a simple normal crossings divisor after a modification) in terms of an *orbifold Kähler metric* up to a logarithmic factor. In Section 3 we give some improvements of earlier results of ours, and state the more precise estimate that we obtain in the hyperkähler case, which implies the estimate needed in Section 2. We also show how the more precise estimate implies part (c) of Conjecture 1.1. This estimate is then proved in Section 4 by showing that the coefficients of the special Kähler metric ω are essentially given by periods of the Abelian varieties which are the fibers of f, and the blowup rate of these periods can be controlled using degenerations of Hodge structures. Lastly, in Section 5 we explain how Theorem 1.2 fits into the SYZ picture of mirror symmetry for hyperkähler manifolds.