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PSEUDOSPECTRAL AND SPECTRAL BOUNDS FOR THE OSEEN VORTICES OPERATOR

BY TE LI, DONGYI WEI AND ZHIFEI ZHANG

ABSTRACT. – In this paper, we solve Gallay's conjecture on the spectral lower bound and pseudospectral bound for the linearized operator of the Navier-Stokes equations in $\mathbb{R}^2$ around rapidly rotating Oseen vortices. This shows that the linearized operator becomes highly non-selfadjoint in the fast rotating limit, and the fast rotation has a strong stabilizing effect on vortices. The main difficulty is to handle the nonlocal part of the linearized operator. By introducing the polar coordinate, the linearized operator can be reduced to a family of one-dimensional operators $\widetilde{\mathcal{H}}_k$ for $|k| \geq 1$. For the case of $|k| \geq 2$, the nonlocal part could be treated as a perturbation by establishing some sharp coercive estimates. The case of $|k| = 1$ is critical in some sense. For this case, the nonlocal part is eliminated by constructing a wave operator. After these reductions, the resolvent estimates can be proved by using the multiplier method.

RÉSUMÉ. – Dans cet article, nous résolvons la conjecture de Gallay sur la borne inférieure spectrale et la borne pseudospectrale pour l'opérateur linéarisé des équations de Navier-Stokes dans $\mathbb{R}^2$ autour des tourbillons Oseen en rotation rapide. Cela montre que l'opérateur linéarisé devient très non autojointif dans la limite de rotation rapide et que la rotation rapide a un fort effet stabilisant sur les tourbillons. La principale difficulté est de gérer la partie non locale de l'opérateur linéarisé. En introduisant la coordonnée polaire, l'opérateur linéarisé peut être réduit à une famille d'opérateurs unidimensionnels $\widetilde{\mathcal{H}}_k$ pour $|k| \geq 1$. Pour le cas de $|k| \geq 2$, la partie non locale pourrait être traitée comme une perturbation en établissant des estimations coercitives précises. Le cas de $|k| \geq 1$ est critique dans un certain sens. Dans ce cas, la partie non locale est éliminée en construisant un opérateur d'onde. Après ces réductions, les estimations de résolution peuvent être prouvées en utilisant la méthode du multiplicateur.

1. Introduction

In this paper, we consider the Navier-Stokes equations in $\mathbb{R}^2$

$$\begin{cases} \partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{1}$$

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where $v(t,x)$ denotes the velocity, $p(t,x)$ denotes the pressure and $\nu > 0$ is the viscosity coefficient. Let $\omega(t,x) = \partial_2 v^1 - \partial_1 v^2$ be the vorticity. The vorticity formulation of (1) takes

$$\partial_t \omega - \nu\Delta\omega + v\cdot\nabla\omega = 0, \quad \omega(0,x) = \omega_0(x). \tag{2}$$

Given the vorticity ω, the velocity can be recovered by the Biot-Savart law

$$v(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy = K_{BS} * \omega. \tag{3}$$

It is well known that the Navier-Stokes Equation (2) has a family of self-similar solutions called Lamb-Oseen vortices of the form

$$\omega(t,x) = \frac{\alpha}{\nu t}\mathcal{G}\Big(\frac{x}{\sqrt{\nu t}}\Big), \quad v(t,x) = \frac{\alpha}{\sqrt{\nu t}} v^G\Big(\frac{x}{\sqrt{\nu t}}\Big), \tag{4}$$

where the vorticity profile and the velocity profile are given by

$$\mathcal{G}(\xi) = \frac{1}{4\pi}e^{-|\xi|^2/4}, \quad v^G(\xi) = \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big).$$

It is easy to see that $\int_{\mathbb{R}^2}\omega(t,x)dx = \alpha$ for any $t > 0$. The parameter $\alpha \in \mathbb{R}$ is called the circulation Reynolds number.

To investigate the long-time behavior of (2), it is convenient to introduce the self-similar variables

$$\xi = \frac{x}{\sqrt{\nu t}}, \quad \tau = \log t,$$

and the rescaled vorticity w and the rescaled velocity u

$$\omega(t,x) = \frac{1}{t}w\Big(\log t, \frac{x}{\sqrt{\nu t}}\Big), \quad v(t,x) = \sqrt{\frac{\nu}{t}}u\Big(\log t, \frac{x}{\sqrt{\nu t}}\Big).$$

Then (w,u) satisfies

$$\partial_\tau w + u\cdot\nabla w = Lw, \tag{5}$$

where the linear operator L is given by

$$L = \Delta + \frac{\xi}{2}\cdot\nabla + 1. \tag{6}$$

For any $\alpha \in \mathbb{R}$, the Lamb-Oseen vortex $\alpha\mathcal{G}(\xi)$ is a steady solution of (5). Gallay and Wayne [14, 15] proved that for any integrable initial vorticity, the long-time behavior of the 2-D Navier-Stokes equations can be described by the Lamb-Oseen vortex. More precisely, for any initial data $w_0 \in L^1(\mathbb{R}^2)$, the solution of (5) satisfies

$$\lim_{\tau\to+\infty}\big\|w(\tau) - \alpha\mathcal{G}\big\|_{L^1(\mathbb{R}^2)} = 0, \quad \alpha = \int_{\mathbb{R}^2} w_0(\xi)d\xi.$$

This result suggests that $\alpha\mathcal{G}$ is a stable equilibrium of (5) for any $\alpha \in \mathbb{R}$. This situation is very similar to the Couette flow $(y,0)$ in a finite channel, which is stable for any Reynolds number [9]. Recently, there are many important works [2, 3, 1, 20, 30] devoted to the study of long-time behavior of the Navier-Stokes(Euler) equations around the Couette flow.

To study the stability of αG, it is natural to consider the linearized equation of (5) around $\alpha\mathcal{G}(\xi)$, which takes as follows

$$\partial_\tau w = (L - \alpha\Lambda)w, \tag{7}$$

where Λ is a nonlocal linear operator defined by

$$\Lambda w = v^G \cdot \nabla w + u \cdot \nabla \mathcal{G} = \Lambda_1 w + \Lambda_2 w, \quad u = K_{BS} * w. \tag{8}$$

The operator $L - \alpha\Lambda$ in the weighted space $Y = L^2(\mathbb{R}^2, \mathcal{G}^{-1}dx)$ defined in Section 2 has a compact resolvent. Thus, the spectrum of $L - \alpha\Lambda$ in Y is a sequence of eigenvalues $\{\lambda_n(\alpha)\}_{n\in\mathbb{N}}$. Moreover, $L - \alpha\Lambda$ is also dissipative, so $\mathrm{Re}\lambda_n(\alpha) \le 0$ for any n, α(see [13] for example). A very important problem is to study how the spectrum changes as $|\alpha| \to +\infty$, which corresponds to the high Reynolds number limit(the most relevant regime for turbulent flows).

The eigenvalues corresponding to the eigenfunctions in the kernel of Λ do not change as α varies. Thus, we introduce two operators $L_\perp$ and $\Lambda_\perp$, which are the restriction of the operators L and Λ to the orthogonal complement of $\ker\Lambda$ in Y respectively. We define the spectral lower bound

$$\Sigma(\alpha) = \inf\Big\{\mathrm{Re}\, z : z \in \sigma\big(-L_\perp + \alpha\Lambda_\perp\big)\Big\} \tag{9}$$

and pseudospectral bound

$$\Psi(\alpha) = \Big(\sup_{\lambda\in\mathbb{R}} \|\big(L_\perp - \alpha\Lambda_\perp - i\lambda\big)^{-1}\|_{Y\to Y}\Big)^{-1}. \tag{10}$$

It is easy to see that $\Sigma(\alpha) \ge \Psi(\alpha)$ for any $\alpha \in \mathbb{R}$. For selfadjoint operators, the spectral and pseudospectral bounds are the same. Since $L - \alpha\Lambda$ is a non-selfadjoint operator, $\Sigma(\alpha)$ and $\Psi(\alpha)$ could be different. Let us mention that the pseudospectra has become an important concept in understanding the hydrodynamic stability [27]. The spectral theory of non-selfadjoint operator is also a current active topic [5, 6, 25, 26].

There are many works devoted to studying $\Sigma(\alpha)$ and $\Psi(\alpha)$. Maekawa [22] proved that $\Sigma(\alpha)$ and $\Psi(\alpha)$ tend to infinity as $|\alpha| \to +\infty$. However, the proof does not provide explicit bounds on $\Sigma(\alpha)$ and $\Psi(\alpha)$. Numerical calculations performed by Prochazka and Pullin [23, 24] indicate that $\Sigma(\alpha) = O(|\alpha|^{\frac{1}{2}})$ as $|\alpha| \to +\infty$. For the rigorous analysis, the main difficulty comes from the nonlocal part Λ_2 of the linearized operator. In [10], Gallagher, Gallay and Nier considered the following toy operator(see also Villani [29, 28]):

$$H_\alpha = -\partial_x^2 + x^2 + i\alpha f(x).$$

Let $\Sigma(\alpha)$ be the infimum of the real part of $\sigma(H_\alpha)$ and $\Psi(\alpha)^{-1}$ be the supremum of the norm of the resolvent of H_α along the imaginary axis. Under the appropriate conditions on f, they proved that $\Sigma(\alpha)$ and $\Psi(\alpha)$ go to infinity as $|\alpha| \to +\infty$, and presented the precise estimate of the growth rate of $\Psi(\alpha)$. Their proof used the hypocoercive method, localization techniques, and semiclassical subelliptic estimates.

For the simplified linearized operator $L - \alpha\Lambda_1$, Deng [7] proved that $\Psi(\alpha) = O(|\alpha|^{\frac{1}{3}})$. The same result was proved by Deng [8] for the full linearized operator restricted to a smaller subspace than $\ker(\Lambda)^\perp$ by using the multiplier method and the Weyl calculus [18].

In this paper, we proved the following conjecture proposed by Gallay [11].

THEOREM 1.1. – *There exists $C > 0$ independent of α so that as $|\alpha| \to +\infty$,*

$$\Sigma(\alpha) \ge C^{-1}|\alpha|^{\frac{1}{2}}, \quad C^{-1}|\alpha|^{\frac{1}{3}} \le \Psi(\alpha) \le C|\alpha|^{\frac{1}{3}}.$$