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PROJECTIONS FROM A VON NEUMANN ALGEBRA ONTO A SUBALGEBRA

BY

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RÉSUMÉ. — Cet article est principalement consacré à la question suivante : soient M, N deux algèbres de Von Neumann avec $M \subset N$. S'il existe une projection complètement bornée $P : N \to M$, existe-t-il automatiquement une projection contractante $\widetilde{P} : N \to M$? Nous donnons une réponse affirmative sous la seule restriction que M soit semi-finie. La méthode consiste à identifier isométriquement l'espace d'interpolation complexe $(A_0, A_1)_\theta$ associé au couple (A_0, A_1) défini comme suit : A_0 (resp. A_1) est l'espace de Banach des n-uples $x = (x_1, \dots, x_n)$ d'éléments de M muni de la norme $\|x\|_{A_0} = \|\sum x_i^* x_i\|_M^{1/2}$ (resp. $\|x\|_{A_1} = \|\sum x_i x_i^*\|_M^{1/2}$) .

ABSTRACT. — This paper is mainly devoted to the following question : let M, N be Von Neumann algebras with $M \subset N$. If there is a completely bounded projection $P : N \to M$, is there automatically a contractive projection $\widetilde{P} : N \to M$? We give an affirmative answer with the only restriction that M is assumed semi-finite. The main point is the isometric identification of the complex interpolation space $(A_0, A_1)_\theta$ associated to the couple (A_0, A_1) defined as follows : A_0 (resp. A_1) is the Banach space of all n-tuples $x = (x_1, \dots, x_n)$ of elements in M equipped with the norm $\|x\|_{A_0} = \|\sum x_i^* x_i\|_M^{1/2}$ (resp. $\|x\|_{A_1} = \|\sum x_i x_i^*\|_M^{1/2}$).

Introduction

This paper is mainly devoted to the following question. Let M, N be von Neumann algebras with $M \subset N$; if there is a completely bounded (*c.b.* in short) projection $P : N \to M$, is there automatically a contractive projection $\widetilde{P} : N \to M$?

We give an affirmative answer with the only restriction that M is assumed semi-finite. At the time of this writing, the case when the

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subalgebra M is a type III factor seems unclear, although this might be not too hard to deduce from our results using crossed product techniques from the Tomita-Takesaki theory with which we are not familiar (see the final remark).

If $N = B(H)$, a positive answer (without any restriction on M) was given in [P1], [P2] (and independently in [CS]). I am grateful to Eberhard KIRCHBERG for mentioning to me that a more general statement might be true. It should be mentioned that the above question seems open if «completely bounded» is replaced by «bounded» in the assumption on the projection P. For more results in this direction, see [P3] and [HP2]. We should recall that, by a classical result of TOMIYAMA [T], every norm one projection P from N onto M necessarily is a conditional expectation and in particular is completely positive. In the second part of the paper we give an interpolation theorem which generalizes a result in [P1], as follows. Let N be a von Neumann algebra equipped with a normal semi-finite faithful trace φ. Let us denote by $L_p(\varphi)$ the noncommutative L_p-space associated to (N, φ) in the usual way. Fix $n \geq 1$. Let us denote by A_0 (resp. A_1) the space N^n equipped with the norms

$$\|(x_1, \ldots, x_n)\|_{A_0} = \Big\| \Big(\sum x_i x_i^* \Big)^{1/2} \Big\|_N,$$

$$\|(x_1, \ldots, x_n)\|_{A_1} = \Big\| \Big(\sum x_i^* x_i \Big)^{1/2} \Big\|_N.$$

We prove in section 2 that the complex interpolation space $(A_0, A_1)_\theta$ is the space N^n equipped with the norm

$$\|(x_1, \ldots, x_n)\|_\theta = \Big\| \sum L_{x_i} R_{x_i^*} \Big\|_{B(L_p(\varphi))}^{1/2}$$

where we have denoted by L_x (resp. R_x) the operator of left (resp. right) multiplication by x on $L_p(\varphi)$, and where $p = \theta^{-1}$. Note that the case $\theta = 0$ corresponds to $L_\infty(\varphi)$ identified with N and $\theta = 1$ corresponds to $L_1(\varphi)$ identified with N_* in the usual way. Again in the particular case $N = B(H)$ this result was proved in [P1].

We refer to [Ta1] for background on von Neumann algebras and to [Pa] for complete boundedness.

We will use several times the following elementary fact.

LEMMA 0.1. — *Let $M \subset N$ be von Neumann algebras. Let $(p_i)_{i \in I}$ be a directed net of projections in M such that, for all x in M, $p_i x p_i$ tends to x in the $\sigma(M, M_*)$ topology. Assume that for each i there is a norm one projection $P_i : N \to p_i M p_i$. Then there is a norm one projection P from N onto M.*

Proof. — Let $\mathcal{U}$ be a nontrivial ultrafilter refining the net. For any x in N, we define

$$P(x) = \lim_{\mathcal{U}} P_i(p_i x p_i)$$

where the limit is in the $\sigma(M, M_*)$ sense. Then $P(x) \in M$ and $\|P(x)\| \leq \|x\|$. Moreover, for any x in M we have

$$P_i(p_i x p_i) = p_i x p_i.$$

Hence $P(x) = x$ for all x in M, and we conclude that P is a projection from N to M. □

1. Projections

The main result of this section is the following.

THEOREM 1.1. — *Let $M \subset N \subset B(H)$ be von Neumann algebras with M semi-finite. If there is a completely bounded (c.b. in short) projection $P : N \to M$, then there is a norm one projection $\widetilde{P} : N \to M$.*

Actually, we use less than complete boundedness, we only need to assume that there is a constant C such that for all $x_1, \ldots, x_n$ in N we have

$$\begin{cases} \left\|\sum P(x_i)^* P(x_i)\right\| \leq C^2 \left\|\sum x_i^* x_i\right\|, \\ \left\|\sum P(x_i) P(x_i)^*\right\| \leq C^2 \left\|\sum x_i x_i^*\right\|. \end{cases} \tag{1.1}$$

The proof is given at the end of this section.

NOTATION. — Let φ be a normal faithful semi-finite trace on a von Neumann algebra N. We denote by $L_2(\varphi)$ the usual associated Hilbert space. For any a in N, we denote by L_a (resp. R_a) the operator of left (resp. right) multiplication by a in $L_2(\varphi)$, i.e. we set for all x in $L_2(\varphi)$

$$L_a x = ax, \quad R_a x = xa.$$

The key lemma in the proof of THEOREM 1.1 is the next statement.

LEMMA 1.2. — *Let N be a semi-finite von Neumann algebra with a normal faithful semi-finite trace φ as above. Consider a finite set $x_1, \ldots, x_n$ in N and assume*

$$\left\|\sum_1^n L_{x_i} R_{x_i^*}\right\|_{B(L_2(\varphi))} \leq 1, \tag{1.2}$$

then there is a decomposition $x_i = a_i + b_i$ with $a_i \in N$, $b_i \in N$ such that

$$\left\|\left(\sum a_i^* a_i\right)^{1/2}\right\| + \left\|\sum b_i b_i^*\right\|^{1/2} \leq 1. \tag{1.3}$$

More generally, the main idea of this paper seems to be the identification of the expression

$$\|(x_1,\ldots,x_n)\| = \Big\|\sum\nolimits_1^n L_{x_i}R_{x_i^*}\Big\|_{B(L_2(\varphi))}^{1/2}$$

with the norm of a simple interpolation space obtained by the complex interpolation method. See section 2 for further details.

COROLLARY 1.3. — *Let N be as in Lemma 1.2 and let M be a finite von Neumann algebra equipped with a normalized finite trace τ. Let $P : N \to M$ be any linear map satisfying* (1.1). *Then for all finite sequences $x_1,\ldots,x_n$ in N we have*

$$\sum_1^n \tau\big(P(x_i)P(x_i)^*\big) = \sum_1^n \tau\big(P(x_i)^*P(x_i)\big) \le C^2\Big\|\sum_1^n L_{x_i}R_{x_i^*}\Big\|_{B(L_2(\varphi))}.$$

Proof. — Assume $\|\sum L_{x_i}R_{x_i^*}\| \le 1$. Let a_i, b_i be as in LEMMA 1.2. Let us denote $\|x\|_2 = (\tau(x^*x))^{1/2}$ for all x in M. Then we have

$$\begin{aligned}\Big\{\sum\|P(x_i)\|_2^2\Big\}^{1/2} &\le \Big\{\sum\|P(a_i)\|_2^2\Big\}^{1/2} + \Big\{\sum\|P(b_i)\|_2^2\Big\}^{1/2}\\ &\le \Big\|\sum P(a_i)^*P(a_i)\Big\|^{1/2} + \Big\|\sum P(b_i)P(b_i)^*\Big\|^{1/2}\\ &\le C. \quad \square\end{aligned}$$

LEMMA 1.4.. — *Let N be as in Lemma 1.2 and let $M \subset N$ be a finite von Neumann subalgebra. Assume that there is a projection $P : N \to M$ satisfying* (1.1). *Then for all nonzero projection p in the center of M and for all unitary operators $u_1,\ldots,u_n$ in M we have*

$$n = \Big\|\sum_1^n L_{pu_i}R_{(pu_i)^*}\Big\|_{B(L_2(\varphi))}. \tag{1.4}$$

Proof. — Fix p as in LEMMA 1.4. By [Ta1, p. 311] there is a finite trace τ on M with $\tau(p) \ne 0$. By COROLLARY 1.3 applied to the normalized trace $x \mapsto \tau(p)^{-1}\tau(x)$ on $pMp = pM$ we have

$$n = \sum \|pu_i\|_2^2 \le C^2\Big\|\sum_1^n L_{pu_i}R_{(pu_i)^*}\Big\|_{B(L_2(\varphi))}.$$

To replace C^2 by 1 in this inequality, we use the same trick as HAAGERUP in [H1]. Let

$$T_n = \sum_1^n L_{pu_i}R_{(pu_i)^*}.$$