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ON THREE-DIMENSIONAL VORTEX PATCHES

PAR

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RÉSUMÉ. — Nous étudions, en dimension trois d'espace, l'existence et la régularité de la solution du système d'Euler incompressible pour une donnée analogue aux poches de tourbillon définies en dimension deux d'espace par A. Majda [7]. Nos résultats sont comparables à ceux obtenus dans le cas bidimensionnel par J.-Y. Chemin [4], mais l'existence de la solution est seulement locale en temps (globale cependant dans le cas axisymétrique).

ABSTRACT. — We study in three space dimensions the existence and smoothness of the solution of the incompressible Euler system for data analogous to the patches of vorticity defined in two space dimensions by A. Majda [7]. Our results are similar to those obtained in the two-dimensional case by J.-Y.Chemin [4], but the existence of the solution is only local in time (global in the axisymmetric case).

Introduction

The movement in $\mathbb{R}^d$ of an ideal incompressible fluid is described by the so-called incompressible Euler system. For this system, the short time existence of a solution of the Cauchy problem with smooth data has been known for a while. In his survey paper [7], MAJDA shows that this elementary result leads to several important problems as the global existence of a solution for smooth data or the (short time or global) existence of a solution for singular data. Here, we consider the Cauchy problem for merely Lipschitzian data.

In the two-dimensional problem, MAJDA [7] introduced *constant patches of vorticity*, which remain such constant patches thanks to a result of YUDOVITCH [9], and asked whether the boundary of such a patch remains

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smooth when it is initially smooth. This question was recently solved by CHEMIN [4], and we also refer to the survey of GÉRARD [6] for an account on recent two-dimensional results.

In this paper, we still consider the problem of patches of vorticity, but for higher space dimensions. Actually we chose the space dimension $d = 3$ for the sake of simplicity, but it is clear that similar results hold when $d > 3$. It is easy to see that, as soon as $d > 2$, compact patches of vorticity cannot be constant patches, and therefore we introduce some spaces of vorticity naturally related to the geometry of compact patches.

Adapting the method of CHEMIN [4], we establish the same results as when $d = 2$, but we get only a short time existence theorem as could be expected. However, our results are also global in time when the initial velocity field is *axisymmetric* as in MAJDA [7]. Finally, we have been informed that a chapter of SERFATI's thesis [8] is also devoted to this problem of multi-D vortex patches, but it is considered there from a Lagrangian point of view.

1. Notation and statement of the main result

1.a. Vectorial notation.

In this paper, we call *vector field* any $\mathbb{R}^3$-valued distribution defined on $\mathbb{R}^3$. The components of the vector field v are denoted by v_1, v_2 and v_3. When the products of the components are well defined, the scalar product of the two vector fields v and w is

$$\langle v, w\rangle = v_1w_1 + v_2w_2 + v_3w_3$$

while their vector and tensor products are respectively :

$$v \wedge w = \begin{pmatrix} v_2w_3 & - & v_3w_2 \\ v_3w_1 & - & v_1w_3 \\ v_1w_2 & - & v_2w_1 \end{pmatrix}, \quad v \otimes w = \begin{pmatrix} v_1w_1 & v_2w_1 & v_3w_1 \\ v_1w_2 & v_2w_2 & v_3w_2 \\ v_1w_3 & v_2w_3 & v_3w_3 \end{pmatrix}.$$

Using the notation $\partial_j = \partial/\partial x_j$ and the formal «vector» ∇ with «components» ∂_1, ∂_2 and ∂_3, the expressions $\langle \nabla, v\rangle$, $\nabla \wedge v$ and $\nabla \otimes v$ defined formally as above will denote respectively the divergence, the curl and the gradient (i.e. the Jacobian matrix) of the vector field v. Similarly, we will use

$$\langle v, \nabla\rangle\, w = \sum_j v_j\partial_j w \quad \text{and} \quad \langle \nabla, v \otimes w\rangle = \sum_j \partial_j(v_jw)$$

which satisfy, when all the products are well defined,

$$\langle \nabla, v \otimes w \rangle = \langle \nabla, v \rangle \, w + \langle v, \nabla \rangle \, w.$$

Finally, in expressing the Biot-Savart law (LEMMA 2.2), we will use the notation :

$$(v * w)(x) = \int v(x-y) \wedge w(y) \,\mathrm{d}y = \int v(y) \wedge w(x-y) \,\mathrm{d}y.$$

With this notation, we can write *the incompressible Euler system*, which provides a model for the movement of a non-viscous liquid in the space $\mathbb{R}^3$, as follows :

$$\begin{cases} \partial_t v + \langle v, \nabla \rangle v = -\nabla p, \\ \langle \nabla, v \rangle = 0, \\ v_{|t=0} = v^0, \end{cases}$$

where the unknown v is a function of the time variable $t \in \mathbb{R}_+$ valued in the space of vector fields (for short, we will say that v *is* a vector field, even when it is time-dependent), and v^0 is a divergence free data belonging to the Lebesgue space L^p for some $1 < p < \infty$. The equation

$$\partial_t v + \langle v, \nabla \rangle v = -\nabla p$$

simply means that the curl of the left side is identically zero, or equivalently, that the left side is the gradient of a scalar distribution $-p \in \mathcal{S}'$, but it can be proved that this distribution p (called the *pressure*) is completely determined by the problem up to a function of t only, and our first step will be to get rid of it.

Our paper discusses existence and uniqueness results for the solution v of this incompressible Euler system. To be able to state precise results, we now introduce the functional spaces where we will look for these solutions.

1.b. Hölder spaces and dyadic analysis.

For all the objects and estimates we describe here, we refer to BONY [2] and CHEMIN [4]. When $s \in \mathbb{R} \setminus \mathbb{Z}_+$ (resp. when $s \in \mathbb{Z}_+$), we denote by C^s (resp. by C^s_*) the Hölder space with exponent s, and in both cases the corresponding norm is denoted by $\|v\|_s$. *The letter r will always denote a real number from the open interval* (0,1), so that the space C^r is the usual space of bounded functions v satisfying

$$\big|v(x) - v(y)\big| \le C|x-y|^r$$

for some constant C and all $x, y \in \mathbb{R}^3$. More generally, if Ω is an open subset of $\mathbb{R}^3$, the space $C^r(\Omega)$, with the norm $\|v\|_{r(\Omega)}$, is the space of all $v \in L^\infty(\Omega)$ satisfying the previous estimate for all $x, y \in \Omega$ (we take this unusual definition — without requiring that x and y stay in the same component of Ω — just to simplify the proofs below : actually, we could have used everywhere the standard definition).

We have the standard interpolation estimate

$$\|v\|_{\mu s+(1-\mu)t} \leq \|v\|_s^\mu \, \|v\|_t^{1-\mu} \quad \text{for} \quad 0 \leq \mu \leq 1,$$

and the L^∞ norm can also be estimated by interpolating between C_*^0 and C^s, $s > 0$: it is the *logarithmic interpolation estimate*

$$\|v\|_{L^\infty} \leq C_s L\big(\|v\|_0, \|v\|_s\big) \quad \text{for} \quad s > 0,$$

where the function

$$L(a,b) = a \operatorname{Log}\Big(2 + \frac{b}{a}\Big)$$

is an increasing function of both variables a and $b \in \mathbb{R}_+$.

When v and w are two Hölder distributions, we denote by $T_v w$ the paraproduct of w by v, for which we have the estimate

$$\|T_v w\|_s \leq C_{s,t} \|v\|_{-t} \, \|w\|_{s+t} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

which is still true for $t = 0$ provided that $\|v\|_{-t}$ is replaced with $\|v\|_{L^\infty}$. When $t > 0$, $v \in C^s$ and $w \in C^{t-s}$, the product of the two distributions v and w is well defined and we have

$$vw = T_v w + T_w v + R(v,w),$$

where the remainder operator R satisfies

$$\big\|R(v,w)\big\|_t \leq C_{s,t} \|v\|_s \, \|w\|_{t-s} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

so that we have the useful estimate

$$\big\|(v - T_v)w\big\|_{\min(s,t)} \leq C_{s,t} \|v\|_s \, \|w\|_{t-s} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

where $\|w\|_{t-s}$ must be replaced with $\|w\|_{L^\infty}$ when $s = t$.

Next, we will say that the pseudodifferential operator $a(D)$ has a *homogeneous symbol* if $a \in C^\infty(\mathbb{R}^3)$ satisfies, for $\mu \geq 1$ and large $\xi \in \mathbb{R}^3$,

$$a(\mu\xi) = \mu^m a(\xi).$$