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SELECTA
YVES MEYER
Volume 1

Edited by A. Bonami, S. Jaffard & S. Seuret

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75 231 Paris Cedex 05, France

Tél : (33) 01 44 27 67 99 • Fax : (33) 01 40 46 90 96
documents_math@smf.emath.fr • http://smf.emath.fr/

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INTRODUCTION

by

Aline Bonami, Stéphane Jaffard & Stéphane Seuret

Yves Meyer is a world famous mathematician and his scientific achievements have been eminently publicized, for example when he received the Gauss and the Abel prizes. So one might wonder whether a selection of his collected works could shed any additional light on them. Actually these prizes focused attention on the role he had played in the birth and elaboration of wavelet analysis, certainly a key step in his scientific trajectory but not one that should shade his many other discoveries. We truly believe that even this major achievement can only be fully appreciated if one is aware of the diversity and depth which characterize the other periods of his scientific trajectory, and how these periods interrelate, in particular leading to the development of wavelet theory.

The purpose of these *Selecta* is to illustrate this evolution and to explain its coherence through two families of articles. First, we have collected and reproduced a series of articles by Yves Meyer, two of them unpublished, which are representative of each of the topics on which he worked. Second, we have supplemented these by companion texts from other mathematicians, who have been collaborating with Yves or who have been influenced by his ideas. Together they put into perspective the wide range of subjects he has considered, from the time he first became interested in them to their current status. They highlight the many breakthroughs he has made and their lasting impact.

All these texts will allow the readers to measure for themselves the impressive influence of Yves Meyer's achievements in various mathematical fields, from number theory to functional and harmonic analysis, and PDEs, to name but a few. It is also striking that this impact often extended to other sciences—a typical example is what is now called "Meyer sets" in crystallography (see the contribution of Denis Gratias and Marianne Quiquandon). Another remarkable example is the revolution that wavelets brought about in mathematical statistics. At the end of the 1980's, David Donoho had developed a framework in order to create mini-max estimators which required a basis that could "work" simultaneously for a large collection of function spaces. More precisely, the functions belonging to these spaces had to be characterized by conditions bearing on the moduli of their coefficients with respect to this basis. It was well known that the Fourier basis could not be used (for example, this property fails for L^p spaces as soon as $p \neq 2$), and the same obstruction shows up for the "classical" bases commonly found in analysis textbooks. David Donoho discussed this problem with Dominique Picard and Gérard Kerkacharian, and they immediately realized that such magic bases were already presented in the book "Ondelettes et opérateurs" (at that time available only in French) which Yves had just published, and which contained a detailed study of the wavelet characterization of Sobolev and Besov spaces. This conversation was the starting point of many key works that changed the field of mathematical statistics at the beginning of the 1990s.

The role of serendipity in the evolution of Yves Meyer's interests has often been mentioned. His meeting in 1984 with Jean Lascoux copying the seminal wavelet paper by Alex Grossmann and Jean Morlet on the copy machine shared with the theoretical physics department at École polytechnique is now part of the folklore of wavelets. It is emblematic of his openness to new ideas and readiness to jump into new subjects. Of other less publicized examples let us mention just one. In 1974, during a visit at the mathematics department of Washington University in Saint Louis, Yves was engaged by the energetic motivation of Raphy Coifman, and persuaded to work on the Calderón conjectures. This initial impulse would lead to a life-lasting collaboration and friendship of two mathematicians whose frames of mind complement each other ideally. This also gave rise to a remarkable mathematical school around Calderón's and Kato's conjectures, as shown in the contributions by Pascal Auscher, Joan Verdera and Raphy Coifman himself.

This book makes it clear that Yves Meyer never rested in the comfort of bounding his thought to a specific subject. Nonetheless, his trajectory has not consisted of random jumps between unrelated areas of mathematics. He has always wanted to be challenged by new questions, attacking new problems with creativity and the extensive scientific knowledge and techniques that he had assimilated from his past achievements. This is made particularly obvious by his work on crystalline measures. This topic is related to one of the very first on which he worked in the 1960s, and recently he has come back to it (see Chapter 1 of this volume), but now with a new approach fed by 60 years of enrichment, which is allowing him to reach results that would not have been accessible in the 1960s.

The scientific career of Yves Meyer is a remarkable interplay between serendipity and coherence. We hope that these volumes will give a rich and multi-faceted understanding of how a great scientific mind developed and flourished, and serve as an example to younger generations of how major scientific advances can be made possible. Yves kindly agreed to write a description of his personal mathematical trajectory, "A Wanderer," for this Selecta.

Let us now describe the content of the three volumes. It should first be emphasized that while Yves wrote several books, these are not reproduced here. Furthermore we had to choose among his numerous articles, preferring ones giving less technical descriptions, some of them published in proceedings of seminars or conferences. In each chapter, several original contributions put Yves's contributions in perspective, explaining their influence in and out of the considered field.

It was natural to start with his first contributions in harmonic analysis, in relation with number theory and their interplay with crystallography. This is the content of the first chapter, which constitutes the first volume. In this area one should also mention his fundamental book "Algebraic Numbers and Harmonic Analysis," which was published in 1972. Some papers related to this book are reproduced here. Many of the tools that Yves created at that time, and in particular the notion of model sets, have proved to be fundamental in the mathematical theory of quasicrystals. Yves came back to the subject later on, so that the first volume contains 13 papers of several different periods, including his most recent articles on crystalline measures. Among the original contributions, one article due to Robert Moody illustrates the links of his work with quasicrystals. Another text by Valérie Berthé and Reem Yassawi describes its influence on the study of discrete dynamical systems arising in computer science. Denis Gratias and Marianne Quiquandon give the point of view of specialists of crystallography. The last contribution, written by Alexander Olevskii and Alexander Ulanovskii, discusses generalized Poisson formulas and crystalline measures.

This chapter also contains one original text by Yves himself, that comes back to the link between crystalline measures and zeta functions and evokes many authors, from various periods of time, such as Hamburger, Kahane, Mandelbrojt, Guinand. It also gives him the occasion to return to his "amours de jeunesse," but in today's language and framework.

In the second volume we have gathered in Chapter 2 some of his achievements on singular integral operators and their applications. This is followed by a chapter containing some of his results on partial differential equations. This represents respectively 13 and 6 articles by Yves Meyer that are reproduced in this volume.

In Chapter 2, the collected papers post-date the start of his collaboration with Raphy Coifman in 1974 around Calderón's program and most are in collaboration with him. Then, the contribution of Joan Verdera is concerned with the boundedness of the Cauchy integral for complex domains, and different proofs and applications, starting from the work of Coifman and Meyer. In parallel, the paper by Pascal Auscher deals with Kato's conjecture, from the fundamental paper of Coifman, Meyer and McIntosh (which deals with the one variable case) to the most recent developments. Raphy Coifman himself comes back with an original article jointly written with Yves. It deals with the Riemann mapping from a simply connected Lipschitz domain of the complex plane into the unit disc. They proved that this mapping depends in an analytic way of the domain. This is a striking result that they obtained in the 1980s but remained isolated and unfinished until now. This paper is complemented by a text of Raphy written especially for this volume. Finally Alexander Volberg bridges the seminal papers of Coifman and Meyer on multipliers and paraproducts with recent results in a multiparameter context with spectacular applications to complex analysis.

Chapter 3 deals with the contributions of Yves Meyer in PDEs. Even if he did not publish many articles in this domain (when compared to the other chapters), he had a widespread influence, as witnessed by Pierre-Louis Lions, through seminar talks, informal discussions and supervisions of many students. Fabrice Planchon explains how Yves and his collaborators introduced new strategies leading to fine estimates, for instance for the div-curl lemma. The text by Jean-Yves Chemin is centered around non smooth pseudo-differential operators as developed by Yves and Raphy Coifman. Enrique Zuazua describes the influence of Yves in control theory and goes further with new results. Finally Pierre-Louis Lions proposes an article on his collaboration with Yves, in particular around the div-curl lemma and its applications.

The third volume is devoted to the spectacular research advances constituted by wavelet theory and related topics, in particular their implications in signal and image processing. These works are those for which Yves Meyer has been most recognized. This volume contains 16 articles written (alone or with collaborators) between 1986 and 2016, as well as texts by some of the main actors in the developments of the wavelet framework, Ingrid Daubechies and Stéphane Mallat, and also by scientists who used them to investigate implications in mathematical analysis and random processes, such as Stéphane Seuret, or in signal or image processing, such as Patrick Flandrin, Antonin Chambolle and Jean-Michel Morel. A paper by Virginia Ajani, Valeria Pettorino and Jean-Luc Starck describes the use of wavelets in the processing of astronomical images. These are domains where the influence of Yves through the books he wrote was also decisive. We already mentioned "Ondelettes et

Opérateurs," later translated in English as "Wavelets and operators," which was one of the very first books on wavelet analysis, and remains today as the one with the most developed mathematical content. "Wavelet methods for pointwise regularity and local oscillation of functions" (in collaboration with Stéphane Jaffard) was an important milestone in the mathematical classification of pointwise singularities and the characterization of "chirp behavior," as was "Wavelets, vibrations and scalings". In the beginning of the 2000's, Yves wrote "Oscillating patterns in image processing and nonlinear evolution equations," in which, in particular, he revisited the famous Rudin–Osher image denoising algorithm. This book opened the way to important advances in image processing. In a completely different spirit, "Wavelets: tools for Science and Technology" (in collaboration with Stéphane Jaffard and Robert Ryan), which was meant for a very wide scientific audience, gave a broad and nontechnical panorama of the many breakthroughs, both in mathematics and in a large variety of sciences, that were made possible by wavelet analysis.

Finally the third volume also contains the list of publications of Yves Meyer.

We address our warmest thanks to Pascal Auscher, Guy David, Basarab Mattei and Hervé Queffelec who helped us during the collecting and editing process, to the authors of the "companion texts," to the numerous referees for their helpful support and contributions, and to the Société Mathématique de France, who constantly encouraged and helped us during this endeavor. Most of all we thank Yves Meyer himself who was present to answer questions, give advices, ... all along the process of conception of this book.

We hope that you will enjoy the discovery of the original papers and the reading of the older ones as much as we appreciated collecting them.

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A WANDERER

by

Yves Meyer

> I am alone. I am walking day and night in the wilderness. I am overwhelmed by the beauty of the forest. I am excited and terrified. I am completely lost when I discover the immense river. I suddenly feel peace. Now I can find my way.

1. A promenade

1.1. Harmonic Analysis and Number Theory (1964–1973). — I was my own supervisor when I was writing my PhD. This was not uncommon in France in the sixties. I studied the multipliers of the Hardy space H^1 and the dual of this Hardy space. My results anticipated some fundamental discoveries by C. Fefferman and E. Stein [**8**]. I extended my research to the study of multipliers of closed ideals of the Wiener algebra. A function f of a real variable belongs to the Wiener algebra $\mathbf{A}$ if f is the Fourier transform of an integrable function g. To every closed set $E \subset \mathbb{R}$ is associated a closed ideal $I(E) \subset \mathbf{A}$ consisting of all functions $f \in \mathbf{A}$ which vanish on E. Is any closed ideal defined by its zero set? That is the problem of spectral synthesis. A multiplier of $I(E)$ is simply a continuous function m on the open set which is the complement of E such that $f \in \mathbf{A}$ implies $mf \in \mathbf{A}$. My results permitted to solve a problem about 'strong Ditkin sets' which was raised by Lennart Carleson [**9**]. When my manuscript was completed and typed by my wife, Anne, I brought it to Jean-Pierre Kahane. Kahane was satisfied with my work and I could obtain a "thèse d'État".

During these three years at Strasbourg (1963–1966), Paul-André Meyer was my mentor and soon became a close friend. I owe him my understanding of probability theory. In collaboration with Paul-André I explored the connections between the Littlewood-Paley decomposition and the theory of martingales. This line of research preluded the beautiful results obtained a few years later by Burkholder and Gundy. Twenty years later Martin Meyer, Paul-André's son, became my PhD student.

After completing my PhD in 1967, I was hired at the mathematical department of the new born "Centre Universitaire d'Orsay" which in 1966 was still a piece of "Université de Paris". After the huge students riots of 1968 the Université de Paris was cut into 13 independent pieces (divide and rule policy) and the Centre Universitaire d'Orsay proudly became "Université Paris-Sud". Banach algebras were studied intensively and almost each week spectacular results were announced and proved at Kahane's "séminaire d'analyse harmonique". I admired Nicolas Varopoulos and his beautiful 'tensor algebras' (1967). After reading *Ensembles parfaits et séries trigonométriques* by Kahane and Salem, I became fascinated by the role of Pisot numbers in harmonic analysis. I found a new approach to a theorem by R. Salem and A. Zygmund concerning sets of uniqueness of trigonometric expansion. A set $E \subset [0, 2\pi]$ is a set of uniqueness if any (formal) trigonometric series which converges to 0 everywhere on $[0, 2\pi] \setminus E$ is identically 0. Let E_θ be the Cantor set constructed with a constant dissection ratio $1/\theta$ (instead of $1/3$ in the usual Cantor set). R. Salem and A. Zygmund proved that E_θ is a set of uniqueness if and only if θ is a Pisot number. I proved that these Cantor type sets E_θ also have the property of spectral synthesis (Carl Herz proved it for the usual Cantor set E_3). This paved the road to my construction of quasicrystals. Let us begin with what is today known as a *Meyer set* (this terminology is due to R. V. Moody). A Delone set $\Lambda \subset \mathbb{R}^n$ is (1) a uniformly discrete set such that (2) there exists a compact set K such that $\Lambda + K = \mathbb{R}^n$. A *Meyer set* Λ is defined in **[10]** by the two following properties: (1) Λ is a Delone set and (2) there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$. Later on J. Lagarias proved the following: If both Λ and $\Lambda - \Lambda$ are Delone sets then Λ is a *Meyer set*. Here is my favorite result: If Λ is a *Meyer set*, if $\theta > 1$ is a real number, and if we have $\theta\Lambda \subset \Lambda$, then θ is a Pisot or a Salem number. The converse is true: for each Pisot or Salem number θ, there exists a *Meyer set* Λ such that $\theta\Lambda \subset \Lambda$. I also unveiled the spectral properties of a narrower class of *Meyer set* which I called model sets. Penrose pavings came a few years later. Soon after some strange patterns observed by D. Shechtman in some chemical alloys were identified to specific model sets. These patterns are named quasicrystals. I was astounded to see the relevance of my model sets in chemistry. Robert V. Moody wrote:

> Initially introduced by Y. Meyer in the context of Diophantine approximation and harmonic analysis in his extraordinary book, model sets have become an important tool in the mathematical study of aperiodic order and quasicrystals.

Model sets Λ also play a role in the theory of mean-periodic functions. Mean-periodic functions were introduced and studied by Jean Delsarte and Jean-Pierre Kahane. Let $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ be an increasing sequence of real numbers. Let us assume that the vector space V_Λ consisting of all finite sums $f(x) = \sum_{-\infty}^{\infty} c_j \exp(i\lambda_j x)$, equipped with the topology of uniform convergence on compact intervals, is not dense in the space V of all continuous functions on the real line. Then one can define the vector space C_Λ as the closure of V_Λ in V. Uniform convergence on the real line would yield almost periodic functions. Then Delsarte and Kahane proved the following: there exists a compact set K such that any $f \in C_\Lambda$ which vanishes on K vanishes identically. A quantitative statement would be the existence of a (positive and finite) weight function ω such that for every x and $f \in C_\Lambda$ we have

$$|f(x)| \leq \omega(x) \sup_{y \in K} |f(y)|. \tag{$*$}$$

We say that Λ is a *coherent set of frequencies* if in $(*)$ the weight ω is a constant. This is equivalent to say that any $f \in C_\Lambda$ is a Bohr almost periodic function. *Model sets are coherent sets of frequencies.* More generally I proved that the growth at infinity of the functions $f \in C_\Lambda$ depends on the Diophantine approximation properties of Λ [**12**]. These remarks apply to vibrating spheres [**13**]. Mean periodic functions in several variables were studied by Alain Yger. Forty five years later I returned to these problems and I proved that a locally finite set $\Lambda \subset \mathbb{R}^n$ is a coherent set of frequencies if and only if Λ satisfies the Bochner property [**20**]. This important result should have been proved much earlier.

I constructed an increasing sequence of integers $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ such that, for any real number α, the sequence $\lambda_n \alpha, n \geq 1$, is equidistributed mod 1 if and only if α is transcendental [**11**]. This result preluded the characterization by G. Rauzy of what he called 'normal sets'.

During my years at Orsay Michel Herman was proving his extraordinary theorem. Here is the result: If ϕ is an orientation preserving diffeomorphism of the circle and if the rotation number α of ϕ satisfies a precise Diophantine property, then ϕ is $\mathcal{C}^\infty$-conjugate to the associated rotation. The linear equation which governs this problem is $f(x+\alpha) - f(x) = g(x)$ where g is given and f is the unknown. Marvellous discussions with Herman led to [**14**].

1.2. Singular integral operators (1974–1984). — In 1974 R. R. Coifman convinced me to attack the famous Calderón's conjectures. Alberto Calderón aimed at constructing an improved pseudo-differential calculus where the smoothness assumptions on the coefficients are minimal. He wrote:

> The aim of this greater generality is to obtain stronger estimates and to prepare the ground for applications to the theory of quasilinear and nonlinear differential equations.

Following Calderón's views, these new operators should be defined as singular integral operators, an approach which is much more flexible than the standard representation of an operator in terms of its symbol. The most famous examples are the Cauchy kernel on a Lipschitz curve or the double layer potential on a Lipschitz surface. Calderón was trying to prove that these operators are bounded on L^2. During a visit to Orsay (1980-1981) Alan McIntosh unveiled the unexpected connection between Calderón's program and Kato's conjectures on the domain of square roots of accretive operators. This deep observation and the theory of multilinear operators developed with R. R. Coifman were seminal in the proof of the boundedness of the Cauchy integral on Lipschitz curves (May 1981). Let me now quote [**8**]:

> I used a new magic trick provided by Alan McIntosh and coming from a world which had been mostly ignored by harmonic analysis people. This new world was familiar to those mathematical physicists who were opening new avenues in operator theory and quantum mechanics. Alan discovered that a very natural conjecture raised by Tosio Kato implies the boundedness of the Cauchy kernel on any Lipschitz curve. This conjecture says that the domain of the square root of a maximal accretive operator coincides with the domain of the sesquilinear form defining this operator. How a conjecture which seems so abstract could be connected with the Cauchy kernel on Lipschitz curves? This is the magic of Calderón's program. At this level of generality Kato's conjecture was untrue but this new perspective reshaped everything and we could prove the boundedness of the Cauchy integral on Lipschitz curves... But this detour by mathematical physics was not needed. Guy David built some new "real variable methods" and deduced "the full theorem" from a partial result obtained by Calderón... The story of the boundedness of the Cauchy integral did not stop there. Indeed in 1995, M. Melnikov and J. Verdera found an extraordinary proof. The starting point is a geometric identity due to Karl Menger and rediscovered by M. Melnikov. Karl Menger (1902-1985) was living in Chicago in these times but his work was not given the attention it deserved. M. Melnikov and J. Verdera cleverly used the Menger curvature and gave us the simplest and the most beautiful proof of the L^2 boundedness of the Cauchy integral on Lipschitz curves. Combining this new approach with some subtle variations on the T(b)-theorem, Guy David proved the Vitushkin conjecture which is a special case of Painlevé's problem on analytic capacity. Finally Xavier Tolsa solved Painlevé's conjecture.

Among the byproducts of this intense mathematical activity, let us mention the solution of the Dirichlet problem in Lipschitz domains by the method of layer potentials (Gregory Verchota), the T(1)-theorem (Guy David and Jean-Lin Journé), the fundamental work by Guy David and Stephen Semmes on singular integral operators on rough surfaces, and the solution of Kato's conjecture about square roots of accretive differential operators (Pascal Auscher, Philippe Tchamitchian, et al.). Guy David, Jean-Lin Journé, Pascal Auscher and Philippe Tchamitchian were my graduate students.

Meanwhile I wished to fill the gap between the theory of pseudo-differential operators and these new singular integral operators introduced by Calderón. I wrote *Au delà des opérateurs pseudo-différentiels* [**3**] in that spirit. This book is co-authored with Raphy Coifman. The new mathematical tools I developed in [**3**] happened to be seminal in the beautiful theory of *paradifferential operators* elaborated by Jean-Michel Bony. Paradifferential calculus can be used in nonlinear PDE's which was exactly Calderón's program. But singular integral operators were no longer needed in Bony's work.

1.3. Control theory. — Let me quote [**17**] :

> It happened the 28th of February, 1984. I was running the Goulaouic-Schwartz seminar after Charles Goulaouic's death. Salah Baouendi had already moved to Purdue. Jacques-Louis Lions was then the head of the French Space Agency (CNES). Lions accepted to give a talk at the seminar. The main issue raised by Lions was the control and stabilization of some large oscillations which might occur on the Space Laboratory and be dangerous for this flexible structure. The problem had to be fixed before launching the satellite and beginning the construction of the International Space Station. Lions was suggesting that one could attenuate and eventually cancel the vibrations by commanding a tiny rocket fixed on the structure. Lions built a mathematical model for addressing this problem and asked us about a solution. To my greatest surprise I succeeded in solving the problem raised by Lions. My proof [**15**] relied on some properties of nonharmonic Fourier series. Soon after Louis Nirenberg found a simpler proof. Finally Lions discovered a third proof.

A reference is J.-L. Lions, *Sur le contrôle ponctuel de systèmes hyperboliques ou de type Petrowski.* Séminaire EDP (Polytechnique) 1983–1984, no. 20. This had a great impact on my scientific life. I understood that some tools I was using in pure mathematics could be efficient in applied problems. This line of research was completed by Stéphane Jaffard.

1.4. Signal and image processing (1984–1993). — Signal and image processing gave me a second scientific life. As it will be detailed below, I could bridge the gap between (1) the atomic decompositions which were used in signal processing by Dennis Gabor and his followers and (2) some other atomic decompositions which were discovered in mathematics by L. Carleson, R. R. Coifman, G. Weiss, et al. Before unveiling this connection, let me begin with defining analysis and atoms in signal processing. In the Webster's dictionary, the word analysis is given the following meaning: *A breaking up of a whole into its parts so as to find out their nature.* In signal or image processing, these parts or simpler entities are named building blocks, time-frequency atoms, or wavelets. Let me explain how building blocks, time-frequency atoms, and wavelets were discovered and why orthonormal wavelet bases can be labelled by suitable partitions of the time-frequency plane.

The story begins in the early thirties when Eugene Wigner (Nobel prize laureate) introduced the *time-frequency* plane together with the famous *Wigner transform.* The Wigner transform $W(t,\omega)$ of a signal $f(t)$ is a function of the time variable t and the frequency variable ω and is providing a description of f which should be both accurate in time and frequency. In other words, for each given value t of the time variable $W(t,\omega)$ should be the value at the frequency ω of the 'instantaneous Fourier transform' of f. But this instantaneous Fourier transform does not exist and cannot be defined as the limit of a short-time Fourier transform when the size of the window tends to 0. An obstruction is given by Heisenberg uncertainty principle. This principle says that we cannot simultaneously measure the time and the frequency. Therefore unfolding a signal in the time-frequency plane without an a priori information on this signal is an impossible task. The Wigner transform is one among infinitely many other solutions. The Wigner transform of a function $f \in L^2(\mathbb{R})$ is defined by

$$W(t,\omega) = \int_{-\infty}^{\infty} \exp(-i\omega\tau) f(t+\tau/2)\overline{f}(t-\tau/2)\, d\tau.$$

The generalization to $f \in L^2(\mathbb{R}^n)$ is straightforward. The Wigner transform of a signal f with finite energy (or of a function belonging to L^2) is the Weyl symbol of the orthogonal projector $P_f : L^2 \mapsto L^2$ on that function. This observation by Hermann Weyl leads to an important remark. Returning to the one dimensional case a family $\psi_\lambda(x)$, $\lambda \in \Lambda$, is an orthonormal basis of $L^2(\mathbb{R})$ if and only if the Wigner transforms Ψ_λ of ψ_λ satisfy the three following conditions:

(i) $\sum_{\lambda\in\Lambda} \Psi_\lambda(t,\omega) = 1$,

(ii) $\int_{\mathbf{R}^2} \Psi_\lambda \overline{\Psi'_\lambda}\, dt\, d\omega = 0$ if $\lambda \neq \lambda'$,

(iii) $\int_{\mathbf{R}^2} |\Psi_\lambda|^2\, dt\, d\omega = 2\pi$.

If these Wigner transforms were the indicator functions of some subsets B_λ, $\lambda \in \Lambda$, of the time-frequency plane, these subsets would form a partition and satisfy $|B_\lambda| = 2\pi$, $\lambda \in \Lambda$. The converse problem reads as follows:

> Let B_λ, $\lambda \in \Lambda$, be a partition of the time-frequency plane into a disjoint union of Heisenberg boxes. Does there exist an 'adapted orthonormal basis' ψ_λ, $\lambda \in \Lambda$?

The basis ψ_λ, $\lambda \in \Lambda$, is adapted to the corresponding Heisenberg boxes B_λ if, for any $\lambda \in \Lambda$, the Wigner transform of ψ_λ is 'almost supported' by B_λ. In other words this Wigner transform is $O(N^{-q}), q = 1, 2, \dots$ outside the dilated boxes $N\, B_\lambda$ as N tends to infinity. This relation between orthonormal bases and coverings of the time-frequency plane paved the way to D. Gabor's work and to my own contribution.

Dennis Gabor (Nobel prize laureate) was familiar with Wigner's ideas and proposed (1945) to decompose speech signals into a linear combination of 'logons' which are aimed at modeling what linguists call phonemes. Today these logons are named Gabor wavelets. Gabor wavelets $w_{k,l}(t) = \exp(-(t-k)^2)\exp(2\pi tl),\ k,l \in \mathbb{Z}$, are associated to a partition of the time-frequency plane with 'disjoint congruent squares' $Q_{k,l}$ defined by

$$Q_{k,l} = \{(t,\omega) \in [k-1/2, k+1/2] \times [2\pi l - \pi, 2\pi l + \pi],\ k,l \in \mathbb{Z}\}.$$

Gabor thought that his wavelets were a basis. Forty years later the scientific landscape changed. We know that Gabor was wrong. The unexpected twist which is needed to fix this problem was suggested in the early eighties by Kenneth Wilson, a Nobel prize laureate. At the same time Henrique Malvar discovered the same recipe independently. Wilson's claim was proved to be true in a beautiful paper by Ingrid Daubechies, Stéphane Jaffard and Jean-Lin Journé. This finding happened to be seminal in the detection of gravitational waves [**5**].

We now leave *time-frequency atoms* and reach time-scale wavelets. In his work on the renormalization of critical phase transitions, K. Wilson (1972) asserted (without proof) the existence of orthonormal wavelet bases of the form $2^{nj/2}\psi(2^j x - k),\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n$, where ψ *belongs to the Schwartz class.* In one dimension the corresponding Heisenberg boxes $B_{j,k,\eta}$ are defined by

$$B_{j,k,\eta} = \{(t,\omega) \in [k2^{-j} - 2^{-j-1}, k2^{-j} + 2^{-j-1}] \times [\eta\pi 2^j, \eta\pi 2^{j+1}], k,j \in \mathbb{Z}, \eta = \pm 1\}.$$

Today we know that $2^n - 1$ such 'mother wavelets' ψ are needed in n dimensions. A few years later (1977) A. Croisier, D. Esteban and C. Galand (IBM Company) designed the famous quadrature mirror filters. Their work concerned the new-born digital telephone. It seemed that time was ripe for constructing time-scale wavelets after Wilson's claims and the discovery of quadrature mirror filters. It did not happen that way, the construction of time-scale wavelets took about ten years and did not follow the main road. The beautiful connection between quadrature mirror filters and orthonormal wavelet bases was discovered much later by Stéphane Mallat.

Here is the story. *We owe time-scale wavelets to Jean Morlet.* In the seventies Jean Morlet was a research scientist working for the Elf-Aquitaine Co. He was using Gabor wavelets to process seismic data. He was puzzled by some artifacts which came out in this processing. He understood that these artifacts were coming from some strong transients in the data. He knew that Fourier analysis is adapted to stationary signals and may perform poorly in other situations. Similarly Gabor wavelets are adapted to quasi-stationary signals. An example of a quasi-stationary signal is given by the tonal component of an audio signal. That is how and why in the late seventies Morlet switched from time-frequency wavelets to *time-scale wavelets.* In his work on non-stationary signals, J. Morlet revived an identity which was previously discovered by A. Calderón in the late fifties. Morlet was unaware of Calderón's work. Morlet could not prove his identity and was advised by Roger Balian to meet Alex Grossmann.

Alex Grossmann was a renowned physicist who has been mainly working in quantum mechanics. He proved the identity which was proposed by Morlet. He did it using the theory of coherent states in quantum mechanics. Grossmann was unaware of K. Wilson's claims. Then Grossmann and Morlet began a fruitful collaboration and promoted a continuous wavelet analysis which yields a highly redundant representation of a signal in the time-frequency plane. Grossmann and Morlet coined the word time-scale plane to stress the special features of their representation.

At about the same time J-O. Strömberg constructed an orthonormal wavelet basis (1981) in which the mother wavelet ψ is smooth (r continuous derivatives) and localized (exponential decay). Strömberg was unaware of K. Wilson's work. He was not interested in signal processing at that time. Strömberg related wavelet expansions to the theory of unconditional bases of Banach spaces. An unconditional basis yields improved stability in numerical analysis. Indeed the quantization noise (real numbers are replaced by digital approximations) does not damage expansions in an unconditional basis. That is why orthonormal wavelet expansions are numerically stable in most function spaces, while Fourier expansions are unstable. This stability is crucially needed in Donoho's work on wavelet shrinkage. Here we are anticipating since Donoho's achievements on denoising appeared in 1993. One wants to recover a signal which has been altered by an additive white noise. Some a priori knowledge on this signal is assumed. This knowledge is given in terms of smoothness or shape. The recipe consists in putting to 0 all the wavelet coefficients whose magnitude is smaller than a small constant depending on the noise level. The other coefficients are moved towards 0 by an amount given by this small constant. This wavelet shrinkage does not alter the a priori knowledge on the signal if wavelets form an unconditional basis of the Banach space which is used in the modeling.

My contribution consisted in organizing these separate findings into a unified theory. In 1984, I related the Grossmann-Morlet wavelets to Calderón's work. Motivated by my joint work with Ingrid Daubechies [**4**] I constructed (1985) an orthonormal wavelet basis of $L^2(\mathbb{R}^n)$ of the form $2^{nj/2}\psi(2^j x - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, where ψ belongs to the Schwartz class. Indeed $2^n - 1$ such functions ψ are needed. The n dimensional case is a joint work with Pierre-Gilles Lemarié [**7**]. K. Wilson's claims were given a scientific status. Then I discovered that these wavelets are the approximate eigenfunctions of many of the generalized Calderón-Zygmund singular integral operators I had constructed with R. R. Coifman. This yielded sparse matrix representations and has far reaching applications in numerical analysis. Finally I stressed the relevance of Strömberg's work.

The digital revolution bloomed in the eighties and efficient algorithms were crucially needed in signal and image processing. The JPEG standard (still image compression) was developed at that time. In 1987 Ingrid Daubechies achieved the outstanding construction of compactly supported wavelets with an assigned regularity. Moreover she designed efficient (i.e. stable) and fast algorithms for computing the corresponding wavelet coefficients.

As it was said above my unified theory was further completed by Stéphane Mallat. In 1986, still being a graduate student, Mallat understood that the quadrature mirror filters designed by Esteban and Galand were paving the road to the construction of orthonormal wavelet bases. Mallat also made the connection with the pyramidal algorithms of Burt and Adelson. This bridged gaps between (1) orthonormal wavelet bases, (2) pyramidal algorithms in image processing, and (3) numerical analysis. Finally A. Cohen proved that generic quadrature mirror filters generate orthonormal wavelets.

Ingrid Daubechies and Albert Cohen designed bi-orthogonal wavelet bases with compact supports. By renouncing to orthogonality they could impose symmetry which is crucial in image processing. These algorithms are used in the new-born standard for still image compression (JPEG2000) and are also present inside existing toolkits which are used in signal or image processing.

Today wavelets and multiscale methods are present in numerical analysis, signal and image processing, and statistics. But the fundamental issue raised by Eugene Wigner is still open. It consists in finding the optimal unfolding of a given signal f in the time-frequency plane. It is what the Wigner transform of f is aimed at doing but does not actually do. Let us describe some tentative work in this direction.

The Yale group around Raphy Coifman proposed an interesting answer to the problem raised by Eugene Wigner. Coifman used a deep observation which was already mentioned: orthonormal bases are labelled by partitions of the time-frequency plane. Here are some examples. Paving the plane with congruent squares leads to Gabor wavelets. More precisely it leads to the Malvar-Wilson basis which is the orthogonal version of the Gabor wavelets. If one is using Heisenberg boxes of the form $[k2^{-j}, (k+1)2^{-j}) \times [\eta\pi 2^j, \eta\pi 2^{j+1})$, $k, j \in \mathbb{Z}$, $\eta = \pm 1$, usual orthonormal wavelet bases are obtained. But infinitely many other choices exist. Among these new bases we find wavelet-packets which yield an adaptive filtering of a given signal $s(t)$. If one is looking for an adaptive segmentation of $s(t)$, local trigonometric bases should be used. The discovery of local trigonometric bases was anticipated by Henrique Malvar who was working on audio signals, and also by Martin Vetterli.

The best basis algorithm was proposed by the Yale group. A library of bases is given. Each 'book' in the library is an orthonormal basis composed of time-frequency atoms. Each 'book' is accessed by a fast algorithm. The best basis algorithm selects the basis in which the representation of a given signal is the 'shortest one'. A similar definition extends to a collection $\mathcal{C}$ of signals equipped with some probability measure μ which tells how likely a given signal is to be found in $\mathcal{C}$. Sparsity is measured by the entropy of the string of coefficients which appears in the expansion and is averaged over μ. This approach paves the road between time-frequency analysis and other best basis algorithms as Karhunen-Loève expansions or Independent Component Analysis (ICA).

Wavelet series expansions have some remarkable mathematical properties which Fourier series expansions cannot offer. Stéphane Jaffard proved that one can evaluate the Hölder exponent of a continuous function f *at a given point* x_0 by size estimates on the wavelet coefficients of f around x_0. The precise statement can be found in [**6**]. Antoni Zygmund was extremely surprised by this theorem. Indeed Fourier series expansions cannot achieve this goal. That is why wavelets are seminal in *multifractal analysis*, a chapter of mathematical physics which was opened by Uriel Frisch and Giorgio Parisi and developed by Jaffard.

An unexpected vindication of wavelet analysis ends this section. *The 14th of September 2015 gravitational waves were detected by the LIGO observatories. The algorithm used in this detection was elaborated by Sergey Klimenko. Time-frequency wavelets are playing a seminal role in Klimenko's algorithm. As it was said above Klimenko's algorithm is based on the remarkable Malvar-Wilson basis constructed by I. Daubechies, Stéphane Jaffard and Jean-Lin Journé* [**5**].

1.5. Navier-Stokes equations (1994–1999). — My interest in Navier-Stokes equations arose from the wavelet revolution and, more precisely, by a question raised by Jacques-Louis Lions. J-L. Lions wanted to know my opinion on an intriguing paper by Guy Battle and Paul Federbush entitled 'Navier and Stokes meet the wavelets'. Lions was not convinced by the paper and asked me for a deeper analysis. This paper was motivated by the following remarks. It was reasonable to believe that wavelet based Galerkin schemes could overcome pseudo-spectral algorithms which were acknowledged as being the best solvers for Navier-Stokes equations. This belief was grounded on some well known observations: Turbulent flows are active over a full range of scales and one is tempted (1) to decouple Navier-Stokes equations as a sequence $E_j, j \in \mathbb{Z}$, of equations where the evolution is confined to a given scale 2^{-j} and (2) to understand the nonlinear interactions between scales and the energy transfers across scales. But the only existing algorithms which permit to travel across scales while keeping an eye on the frequency contents are the Littlewood-Paley expansion or the wavelet analysis. Furthermore micro-local analysis and

Littlewood-Paley expansions have been successfully applied to Navier-Stokes or Euler equations by Jean-Yves Chemin and his students. Times were ripe for replacing Littlewood-Paley analysis by fast numerical schemes which have the same scientific contents, i.e. by wavelet analysis.

But this endeavour was not a success story. As often in science, something else was found. Marco Cannone made two main discoveries. He proved that Littlewood-Paley expansions were more effective than the wavelet expansions used in the Battle-Federbush paper. He then observed that a strategy due to Fujita and Kato but also used by Cazenave and Weissler was even more effective. Once this was clarified, Marco Cannone and Fabrice Planchon improved on the Fujita-Kato theorem. Indeed they proved global existence for solutions $u(x,t) \in C([0,\infty), L^3(\mathbb{R}^3))$ to the Navier-Stokes equations whenever the initial condition u_0 is oscillating. Uniqueness was proved by Pierre-Gilles Lemarié a few years later. The oscillating character of u_0 is defined by the smallness of a Besov norm in a suitable Besov space $B_p, 3 \leq p < \infty$, the precise bound depending on p. An equivalent condition is given by simple size estimates of the wavelet coefficients. These methods did not apply to the limiting case $p = \infty$. The best result in this direction was finally obtained by Herbert Koch and Daniel Tataru. M. Cannone, Pierre-Gilles Lemarié, and F. Planchon prepared their PhD under my direction.

1.6. The div-curl lemma. — In 1989 I was a professor at Université Paris-Dauphine. One day Pierre-Louis Lions jumped into my office, happy and energetic and asked me if the following fact could be true: Let $E = (E_1, \dots, E_n)$ and $R = (R_1, \dots, R_n)$ be two vector fields belonging to $(L^2(\mathbb{R}^n))^n$. Let us assume that the divergence of E is identically 0 and that the curl of R is also identically 0. Then Pierre-Louis Lions claimed that the pointwise inner product $f(x) = E(x) \cdot R(x) = E_1(x)R_1(x) + \cdots + E_n(x)R_n(x)$ should belong to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. I was skeptical and answered to Pierre-Louis "If it were true, I would know it". My answer was ridiculous since I proved Pierre-Louis's claim during the night. Other proofs were discovered by R. Coifman and S. Semmes [**1, 2**]. We have more: Let us assume now that E belongs to L^p and R to L^q where $1 < p < \infty$ and $1/p + 1/q = 1$. Then the same hypothesis on the divergence of E and on the curl of B implies that $E(x) \cdot R(x)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. This is striking since each product $E_j(x)R_j(x)$ belongs to L^1 but certainly not to $\mathcal{H}^1(\mathbb{R}^n)$. There is a hidden cancellation occurring in the sum. When I could prove this theorem I immediately sent the proof to Eli Stein. He was delighted. This theorem is deeply related to the famous div-curl lemma of F. Murat and L. Tartar (1978). The remarkable paper *On the product of functions in* BMO *and* H^1 by Aline Bonami, Tadeusz Iwaniec, Peter Jones, and Michel Zinsmeister belongs to this line of research.

1.7. Improved Sobolev embeddings (2000–2004). — My interest in nonlinear evolution equations oriented my research in another direction. The above mentioned results let me hope that a functional norm could govern the eventual blowup of Navier-Stokes equations. This endeavour opened a program which was completed by some unexpected improvements on Gagliardo-Niremberg inequalities. For a number of nonlinear evolution equations, blowup may happen even if the initial condition is smooth and compactly supported. The new Gagliardo-Nirenberg inequalities improve our understanding of the occurrence of blow-up for solutions of the nonlinear heat equation. More precisely these new Gagliardo-Nirenberg inequalities explain why there is no blowup when the initial condition is oscillating. Our approach does not yield new results here but places classical facts in a new perspective.

This line of research led to important results by Albert Cohen, Ingrid Daubechies, Michel Ledoux and Ron DeVore.

A reference is [**16**].

1.8. Image processing. — My work in image processing stems from my friendship with Jean-Michel Morel. Morel is suggesting that I was his mentor when we were together at Université Paris-Dauphine. My enthusiasm for signal processing would have been contagious and have led him to image processing. Then he built his extraordinary school. I am feeling quite the opposite and I can assert that Morel was my mentor and my guide. In the Greek mythology, Mentor was the tutor of Telemachus. In [**16**] I proposed a new model for still image processing. It is a refinement on a celebrated model proposed by S. Osher, L. Rudin, and E. Fatemi. It amounts to see textures as oscillating patterns and to use some adapted functional spaces to model such textures. This line of research was followed by V. Caselles, A. Chambolle, L. Vese, and by my students Jérôme Gilles and Ali Haddad.

1.9. Crystalline measures, quasicrystals and irregular sampling. — Since 2010 I am collaborating with Alexander Olevskii. It is delightful. We were addressing some intriguing issues on irregular sampling. This line of investigation has some unexpected connections with quasicrystals and with the work of Andrew Guinand in number theory. I could prove some intriguing results announced by Guinand [**18, 19**]. I found a new proof of a beautiful theorem by Hans Hamburger relating quasicrystals to zeta functions [**21**].

I proved that a locally finite set $\Lambda \subset \mathbb{R}^n$ is a coherent set of frequencies if and only if Λ has the Bochner property [**20**]. The Bochner property is the following assertion: for any Radon measure μ on the Bohr compactification of $\mathbf{R}^n$ there exists a bounded Radon measure ν on $\mathbb{R}^n$ such that the Fourier transforms of μ and ν coincide on Λ.

2. Key Career Highlights

Appointments

1999–2008: Professor at École Normale Supérieure de Cachan
1995–1999: Full research position at CNRS
1985–1995: Professor at Université Paris-Dauphine
1980–1986: Professor at École polytechnique
1966–1980: Professor at Université Paris-Sud
1963–1966: Teaching assistant at Université de Strasbourg
1960–1963: High school teacher
1957–1960: École Normale Supérieure (Ulm)

Scientific distinctions. — Peccot Prize (1969), Salem Prize (1970), Carrière Prize (1972), Grand Prix de l'Académie des Sciences (1984), Gauss Prize (2010), Abel Prize (2017), Membre de l'Académie des sciences (France), Foreign Honorary Member of the American Academy of Arts & Sciences, Foreign Associate of the National Academy of Sciences, Member of the Norwegian Academy of Science and Letters, Doctor honoris causa of Universidad Autonoma de Madrid, Doctor honoris causa of EPFL.

Ph.D. students supervised by Yves Meyer. — Aline Bonami, Noel Lohoué, Jean-Pierre Schreiber, Michel Bruneau (codir. P-A. Meyer), Marc Frisch, Françoise Piquard (codir. N. Varopoulos), Gérard Bourdaud, Alain Yger, Michel Zinsmeister, Jean-Lin Journé, Guy David, Michel Emsalem, François Gramain, Jean-Paul Allouche, Miguel Escobedo Martinez (codir. H. Brezis), Pierre-Gilles Lemarié, Martin Meyer, Philippe Tchamitchian, Chantal Tran-Oberlé, Mohamed El Hodaibi, Oscar Barraza, Yang Qi Xiang, Ramzi Labidi, Pascal Auscher, Stéphane Jaffard, Albert Cohen, Salifou Tembely, Sylvia Dobyinski, Sylvain Durand, Taoufik El Bouayachi, Marco Cannone, Freddy Paiva, Khalid Daoudi (codir. J. Levy-Véhel, INRIA), Abderrafiaa El-Kalay, Hong Xu, Fabrice Planchon, Fatma Trigui, Patrick Andersson, Antoine Ayache, Henri Oppenheim, Mehdi Abouda, Guillaume Bernuau, Frédéric Oru, Lorenzo Brandolese, Soulaymane Korry (codir. B. Maurey), Ali Haddad, Diego Chamorro, Jérôme Gilles, Xiaolong Li.

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