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# SPRINGER FIBER COMPONENTS IN THE TWO COLUMNS CASE FOR TYPES $A$ AND $D$ ARE NORMAL

BY NICOLAS PERRIN & EVGENY SMIRNOV

ABSTRACT. — We study the singularities of the irreducible components of the Springer fiber over a nilpotent element $N$ with $N^2 = 0$ in a Lie algebra of type $A$ or $D$ (the so-called two columns case). We use Frobenius splitting techniques to prove that these irreducible components are normal, Cohen–Macaulay, and have rational singularities.

## 1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic not equal to 2. Let $N$ be a nilpotent element in a Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ (type $A$) or $\mathfrak{g} = \mathfrak{so}_{2n}$ (type $D$). We consider the Springer fiber $\mathcal{F}_N$ over $N$. It is the fiber of the famous Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ over $N$.

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This resolution can be constructed as follows. Let $\mathscr{F}$ be the variety of complete flags in $\mathbb{K}^n$ (resp. $\mathrm{O}\mathscr{F}$ the variety of complete *isotropic* flags, see Section 3 for the description of the Springer fiber $\mathrm{O}\mathscr{F}_N$ in this case). A flag $f = (V_i)_{i\in[0,n]}$, where $V_i$ is a vector subspace of $\mathbb{K}^n$ of dimension $i$, is stabilized by $N \in \mathscr{N}$ if $N(V_i) \subset V_{i-1}$ for all $i > 0$. We shall denote this by $N(f) \subset f$. Define the variety

$$\widetilde{\mathscr{N}} = \{(f, N) \in \mathscr{F} \times \mathscr{N} \ \mid N(f) \subset f\}.$$

The projection $\widetilde{\mathscr{N}} \to \mathscr{F}$ is a smooth morphism thus $\widetilde{\mathscr{N}}$ is smooth. The natural projection $\widetilde{\mathscr{N}} \to \mathscr{N}$ is birational and proper. It is a resolution of singularities for $\mathscr{N}$ called the *Springer resolution.*

The *Springer fibers*, i.e., the fibers of the Springer resolution, are of great interest. They are connected (this can be seen directly or follows from the normality of the nilpotent cone $\mathscr{N}$), equidimensional, but not irreducible. There is a natural combinatorial framework to describe them: Young diagrams and standard tableaux.

The irreducible components of the Springer fibers are not well understood. For example, it is known that in general the components are singular but there is no general description of the singular components. There are only partial answers in type $A$. First, it is known in the so-called *hook* and *two lines* cases that all the components are smooth (see [8]). The first case where singular components appear is the *two columns* case. A description of the singular components in the two columns case has been given by L. Fresse in [5] and [4]. In their recent work [6] L. Fresse and A. Melnikov describe the Young diagrams for which all irreducible components are smooth.

In this paper, we focus on the the two columns case, that is to say, the case of nilpotent elements $N$ of order 2. The corresponding Young diagram $\lambda = \lambda(N)$ has two columns. We want to understand the type of singularities appearing in a component of the Springer fiber.

Let $X$ be an irreducible component of the Springer fiber $\mathscr{F}_N$, resp. $\mathrm{O}\mathscr{F}_N$, in type $A$, resp. $D$, with $N$ nilpotent such that $N^2 = 0$. In the two columns case, we describe a resolution $\pi\colon \widetilde{X} \to X$ of the irreducible component $X$. We use this resolution to prove, for $\mathrm{Char}(\mathbb{K}) > 0$, that $X$ is Frobenius split, and deduce the following result for arbitrary characteristic:

THEOREM 1.1. — *The irreducible component $X$ is normal.*

We are able to prove more on the resolution $\pi$. Recall that a proper birational morphism $f : X \to Y$ is called a rational resolution if $X$ is smooth and if the equalities $f_*\mathscr{O}_X = \mathscr{O}_Y$ and $R^i f_*\mathscr{O}_X = R^i f_*\omega_X = 0$ for $i > 0$ are satisfied. We prove the following

THEOREM 1.2. — *The morphism $\pi$ is a rational resolution.*

Corollary 1.3. — *The irreducible component $X$ is Cohen–Macaulay with dualizing sheaf $\pi_*\omega_{\widetilde{X}}$.*

Rational singularities are well defined in characteristic zero. In this case we obtain the following

Corollary 1.4. — *If* $\mathrm{Char}(\mathbb{K}) = 0$*, then $X$ has rational singularities.*

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## 2. Irreducible components of Springer fibers in type $A$

**2.1. General case.** — Let $N \in \mathfrak{gl}_n$ be a nilpotent element, and let $(m_i)_{i\in[1,r]}$ be the sizes of Jordan blocks of $N$. To $N$ we assign a Young diagram $\lambda = \lambda(N)$ of size $n$ with rows of lengths $(m_i)_{i\in[1,r]}$. We refer to [7] for more details on Young diagrams.

Definition 2.1. — A *standard Young tableau* of shape $\lambda$ is a bijection $\tau\colon \lambda \to [1, n]$ such that the numbers assigned to the boxes in each row are decreasing from left to right, and the numbers in each column are decreasing from top to bottom.

Remark 2.2. — Usually, one requires that the integers in the boxes of a standard tableau *increase*, not decrease from left to right and from top to bottom. However, using *decreasing* tableaux in our case simplifies the notation, so we decided to follow this (rather unusual) definition.

Remark that the datum of a standard tableau $\tau$ is equivalent to the datum of a chain of decreasing Young diagrams $\lambda = \lambda^{(0)} \supset \lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)} = \varnothing$, where $\lambda^{(i)}$ is the set of the $n - i$ boxes with the largest numbers, that is, $\tau^{-1}(\{i + 1, \ldots, n\})$.

Let $f = (V_i) \in \mathscr{F}$ be an $N$-stable flag. We assign to it a standard tableau of shape $\lambda = \lambda(N)$ in the following way. Consider the quotient spaces $V^{(i)} = V/V_i$. The endomorphism $N$ induces an endomorphism of each of these quotients $N^{(i)}\colon V^{(i)} \to V^{(i)}$. Take the Young diagram $\lambda^{(i)}$ corresponding to $N^{(i)}$; it consists of $n-i$ boxes. Clearly, $\lambda^{(i)}$ differs from $\lambda^{(i-1)}$ by one corner box. So we obtain a chain of decreasing Young diagrams, which is equivalent to a standard Young tableau $\tau(f)$.

Let $\tau$ be a standard Young tableau of shape $\lambda(N)$. Define

$$X_\tau^0 = \{f \in \mathcal{F}_N \mid \tau(f) = \tau\}.$$

The following theorem is due to Spaltenstein [15].

Theorem 2.3. — *For each standard tableau $\tau$, the subset $X_\tau^0$ is a smooth irreducible subvariety of $\mathcal{F}_N$. Moreover, $\dim X_\tau^0 = \dim \mathcal{F}_N$, so $X_\tau = \overline{X_\tau^0}$ is an irreducible component of $\mathcal{F}_N$. Any irreducible component of $\mathcal{F}_N$ is obtained in this way.*

**2.2. Two columns case.** — In this paper, we focus on the case of nilpotent elements $N$ such that $N^2 = 0$. This is equivalent to saying that the Young diagram $\lambda(N)$ consists of (at most) two columns. Denote by $r$ the rank of $N$ or, equivalently, the number of boxes in the second column. Let $X = X_\tau$ be the irreducible component of the Springer fiber over $N$ corresponding to a standard tableau $\tau$. Denote the increasing sequence of labels in the second column of the standard tableau $\tau$ by $(p_i)_{i\in[1,r]}$. Set $p_0 = 0$ and $p_{r+1} = n+1$.

According to F.Y.C. Fung [8], the previous Theorem can be reformulated as follows.

Proposition 2.4. — *The irreducible component $X$ is the closure of the variety*

$$X^0 = \left\{ (V_i)_{i\in[0,n]} \in \mathcal{F}_N \;\middle|\; \begin{array}{l} V_i \subset V_{i-1} + \mathrm{Im} N \text{ for } i \in \{p_1, \cdots, p_r\} \\ V_i \not\subset V_{i-1} + \mathrm{Im} N \text{ otherwise} \end{array} \right\}.$$

An easy interpretation of this result is the following

Corollary 2.5. — *The irreducible component $X$ is the closure of the variety*

$$X^0 = \left\{(V_i)_{i\in[0,n]} \in \mathcal{F}_N \;\mid \dim(\mathrm{Im}N \cap V_i) = k \text{ for all } k \in [0,r] \text{ and all } i \in [p_k, p_{k+1})\right\}.$$

*Proof.* We prove this by induction on $i$. We have $\dim(\mathrm{Im}N \cap V_0) = 0$. The result is implied by the following equivalence: $(V_{i+1} \subset V_i + \mathrm{Im}N) \Leftrightarrow (\dim(\mathrm{Im}N \cap V_{i+1}) = \dim(\mathrm{Im}N \cap V_i) + 1)$. □