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EXPLICIT TEICHMÜLLER CURVES WITH COMPLEMENTARY SERIES

by Carlos Matheus & Gabriela Weitze-Schmithüsen

Abstract. — We construct an explicit family of arithmetic Teichmüller curves $\mathscr{C}_{2k}$, $k \in \mathbb{N}$, supporting $\mathrm{SL}(2,\mathbb{R})$-invariant probabilities μ_{2k} such that the associated $\mathrm{SL}(2,\mathbb{R})$-representation on $L^2(\mathscr{C}_{2k}, \mu_{2k})$ has complementary series for every $k \geq 3$. Actually, the size of the spectral gap along this family goes to zero. In particular, the Teichmüller geodesic flow restricted to these explicit arithmetic Teichmüller curves $\mathscr{C}_{2k}$ has arbitrarily slow rate of exponential mixing.

Résumé (*Courbes de Teichmüller explicites avec série complémentaires*)

On construit une famille explicite de courbes de Teichmüller arithmétiques $\mathscr{C}_{2k}$, $k \in \mathbb{N}$, supportant des probabilités $\mathrm{SL}(2,\mathbb{R})$-invariantes μ_{2k} telles que la $\mathrm{SL}(2,\mathbb{R})$-representation associée sur $L^2(\mathscr{C}_{2k}, \mu_{2k})$ a des séries complémentaires pour tout $k \geq 3$. En fait, la taille du trou spectral de cette famille tend vers zéro. En particulier, le flot géodésique de Teichmüller restreint à ces courbes de Teichmüller explicites $\mathscr{C}_{2k}$ a une vitesse de mélange exponentiel arbitrarement lente.

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1. Introduction

Let $\mathcal{H}_g$ be the moduli space of unit area Abelian differentials on a genus $g \geq 1$ Riemann surface. This moduli space is naturally stratified by prescribing the list of orders of zeroes of Abelian differentials: more precisely,

$$\mathcal{H}_g = \bigcup_{\kappa=(k_1,\dots,k_\sigma)} \mathcal{H}(\kappa)$$

where $\mathcal{H}(\kappa)$ is the set of unit area Abelian differentials with zeroes of orders $k_1, \dots, k_\sigma$. Here, we have the constraint $\sum_{j=1}^{\sigma} k_j = 2g-2$ coming from Poincaré-Hopf formula. The terminology "stratification" here is justified by the fact that $\mathcal{H}(\kappa)$ is the subset of unit area Abelian differentials of the complex orbifold of complex dimension $2g+\sigma-1$ of Abelian differentials with list of orders of zeroes κ. See [13, 20, 21, 22] for further details.

In general, the strata $\mathcal{H}(\kappa)$ are not connected but the complete classification of their connected components was performed by M. Kontsevich and A. Zorich [11]. As a by-product of [11], we know that every stratum has 3 connected components at most (and they are distinguished by certain invariants).

The moduli space $\mathcal{H}_g$ is endowed with a natural $\mathrm{SL}(2,\mathbb{R})$-action such that the action of the diagonal subgroup $g_t := \mathrm{diag}(e^t, e^{-t})$ corresponds to the so-called Teichmüller geodesic flow (see e.g., [22]).

After the seminal works of H. Masur [13] and W. Veech [20], we know that any connected component $\mathcal{C}$ of a stratum $\mathcal{H}(\kappa)$ carries a unique $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure $\mu_{\mathcal{C}}$ which is absolutely continuous with respect to the Lebesgue measure in local (period) coordinates. Furthermore, this measure is ergodic and mixing with respect to the Teichmüller flow g_t. In the literature, this measure is sometimes called *Masur-Veech measure*.

For Masur-Veech measures, A. Avila, S. Gouëzel and J.-C. Yoccoz [3] established that the Teichmüller flow is exponential mixing with respect to them. Also, using this exponential mixing result and M. Ratner's work [16] on the relationship between rates of mixing of geodesic flows and *spectral gap* property of $\mathrm{SL}(2,\mathbb{R})$-representations, they were able to deduce that the $\mathrm{SL}(2,\mathbb{R})$ representation $L^2(\mathcal{C}, \mu_{\mathcal{C}})$ has spectral gap.

More recently, A. Avila and S. Gouëzel [2] were able to extend the previous exponential mixing and spectral gap results to general *affine* $\mathrm{SL}(2,\mathbb{R})$-invariant measures.[1]

[1] By affine measure we mean that it is supported on a locally affine (on period coordinates) suborbifold and its density is locally constant in affine coordinates. Conjecturally, all $\mathrm{SL}(2,\mathbb{R})$-invariant measures on $\mathcal{H}_g$ are affine, and, as it turns out, this conjecture was recently proved by A. Eskin and M. Mirzakhani [7].

Once we know that there is spectral gap for these representations, a natural question concerns the existence of *uniform* spectral gap. For Masur-Veech measures, this was informally conjectured by J.-C. Yoccoz (personal communication) in analogy with Selberg's conjecture [19]. On the other hand, as it was recently noticed by A. Avila, J.-C. Yoccoz and the first author during a conversation, one can use a recent work of J. Ellenberg and D. McReynolds [6] to produce $\mathrm{SL}(2,\mathbb{R})$-invariant measures supported on the $\mathrm{SL}(2,\mathbb{R})$-orbits of *arithmetic Teichmüller curves* (i.e., *square-tiled surfaces*) along the lines of Selberg's argument to construct non-congruence finite index subgroups Γ of $\mathrm{SL}(2,\mathbb{Z})$ with arbitrarily small spectral gap. We will outline this argument in Appendix A.

However, it is not easy to use the previous argument to exhibit *explicit* examples of arithmetic Teichmüller curves with arbitrarily small spectral gap. Indeed, as we're going to see in Appendix A, the basic idea to get arithmetic Teichmüller curves with arbitrarily small spectral gap is to appropriately choose a finite index subgroup $\Gamma_2(N)$ of the principal congruence subgroup Γ_2 of level 2 so that $\mathbb{H}/\Gamma_2(N)$ corresponds to N copies of $\mathbb{H}/\Gamma_2$ arranged *cyclically* in order to slow down the rate of mixing of the geodesic flow (since to go from the 0th copy to the $[N/2]$th copy of $\mathbb{H}/\Gamma_2$ it takes a time $\sim N$). In this way, it is not hard to apply Ratner's work [16] to get a bound of the form $1 \lesssim Ne^{-\sigma(N)\cdot N}$ where $\sigma(N)$ is the size of the spectral gap of $\mathbb{H}/\Gamma_2(N)$. Hence, we get that the size $\sigma(N)$ of the spectral gap decays as $\lesssim \ln(N)/N$ along the family $\mathbb{H}/\Gamma_2(N)$. Thus, we will be done once we can realize $\mathbb{H}/\Gamma_2(N)$ as an arithmetic Teichmüller curve, and, in fact, this is always the case by the work of J. Ellenberg and D. McReynolds [6]: the quotient $\mathbb{H}/\Gamma$ can be realized by an arithmetic Teichmüller curve whenever Γ is a finite index subgroup of Γ_2 containing $\{\pm Id\}$ (such as $\Gamma_2(N)$). In principle, this could be made explicit, but one has to pay attention in two parts of the argument: firstly, one needs to derive explicit constants in Ratner's work (which is a tedious but straightforward work of bookkeeping constants); secondly, one needs rework J. Ellenberg and D. McReynolds article to the situation at hand (i.e., trying to realize $\mathbb{H}/\Gamma_2(N)$ as an arithmetic Teichmüller curve). In particular, since the spectral gap decays slowly ($\lesssim \ln(N)/N$) along the family $\mathbb{H}/\Gamma_2(N)$ and the Ellenberg-McReynolds construction involves taking several branched coverings, even exhibiting a single arithmetic Teichmüller curve with complementary series demands a certain amount of effort.

In this note, we propose an alternative way of constructing arithmetic Teichmüller curves with arbitrarily small spectral gap. Firstly, instead of getting a small spectral gap from slow of mixing of the geodesic flow, a sort of "dynamical-geometrical" estimate, we employ the so-called *Buser inequality* to get small spectral gap from the Cheeger constant, a more "geometrical" constant measuring the ratio between the length of separating multicurves and the area of the regions bounded by these multicurves on the arithmetic Teichmüller curve. As a by-product of this procedure, we will have a family $\Gamma_6(2k)$ of finite index subgroups of Γ_6 (the level 6 principal congruence subgroup) also obtained by a cyclic construction such that the size of the spectral gap of $\mathbb{H}/\Gamma_6(2k)$ decays as $\lesssim 1/k$ (where the implied constant can be computed effectively). Secondly, we combine some parts of Ellenberg-McReynolds methods [6] with the ones of the second author [18] to explicitly describe a family of arithmetic Teichmüller curves birational to a covering of $\mathbb{H}/\Gamma_6(2k)$ (that is, the *Veech group* of the underlying square-tiled surface is a subgroup $\Gamma_6(2k)$). As a by-product of this discussion, we show the following result:

Theorem 1.1. — *Suppose that $k \geq 3$.*

i) *For any origami whose Veech group Γ is a subgroup of $\pm\Gamma_6(2k)$, its Teichmüller curve exhibits complementary series and the spectral gap of the regular representation associated to $\mathbb{H}/\Gamma$ is smaller than $1/k$.*
ii) *The Veech group of the origami Z_k (defined in Definition 4.1) of genus $48k+3$ and $192k$ squares is contained in $\pm\Gamma_6(2k)$. In particular, its Teichmüller curve $\mathscr{C}_{2k}$ exhibits complementary series and this family of origamis gives an example that there is no uniform lower bound for the spectral gaps associated to Teichmüller curves.*

We organise this note as follows. In Section 2, we present the cyclic construction leading to the family of finite index subgroups $\Gamma_6(2k)$, $k \in \mathbb{N}$, of Γ_6. We show that the size of the spectral gap of $\mathbb{H}/\Gamma_6(2k)$ decays as $\lesssim 1/k$. In a nutshell, we consider the genus 1 curve $\mathbb{H}/\Gamma_6$, cut along an appropriate closed geodesic, and glue cyclically $2k$ copies of $\mathbb{H}/\Gamma_6$. This will produce the desired family $\mathbb{H}/\Gamma_6(2k)$ because the multicurves consisting of the two copies of our closed geodesic at the 0th and kth copies of $\mathbb{H}/\Gamma_6$ divide $\mathbb{H}/\Gamma_6(2k)$ into two parts of equal area, see Figure 3. Thus, since the area of $\mathbb{H}/\Gamma_6(2k)$ grows linearly with k while the length of the multicurves remains bounded, we'll see that the Cheeger constant decays as $\lesssim 1/k$, and, a fortiori, the size of the spectral gap decays as $\lesssim 1/k$ by Buser inequality. Proposition 2.8 and Remark 2.9 then show i) of Theorem 1.1. In Section 3, we describe an explicit family of square-tiled surfaces whose Veech group is $\mathrm{SL}(2,\mathbb{Z})$. Then, we give a condition that coverings of them have a Veech group which is contained in $\pm\Gamma_6(2k)$, i.e., its Teichmüller curve is birational to a covering of $\mathbb{H}/\Gamma_6(2k)$. These two sections can be read independently from each other. Finally, in the last section, we prove ii) of Theorem 1.1 by constructing explicit origamis which satisfy the given conditions. In particular, as our "smallest" example, we construct an origami with 576 squares whose Veech group is a subgroup of $\Gamma_6(6)$ and, *a fortiori*, whose Teichmüller curve exhibits complementary series (see Corollary 4.3).