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RANDOM SURFACES

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RANDOM SURFACES

Scott Sheffield

Abstract. — We study the statistical physical properties of (discretized) “random surfaces,” which are random functions from $\mathbb{Z}^d$ (or large subsets of $\mathbb{Z}^d$) to E , where E is $\mathbb{Z}$ or $\mathbb{R}$. Their laws are determined by convex, nearest-neighbor, gradient Gibbs potentials that are invariant under translation by a full-rank sublattice $\mathcal{L}$ of $\mathbb{Z}^d$; they include many discrete and continuous height function models (e.g., domino tilings, square ice, the harmonic crystal, the Ginzburg-Landau $\nabla\phi$ interface model, the linear solid-on-solid model) as special cases.

We prove a *variational principle* – characterizing gradient phases of a given slope as minimizers of the specific free energy – and an empirical measure large deviations principle (with a unique rate function minimizer) for random surfaces on mesh approximations of bounded domains. We also prove that the surface tension is strictly convex and that if u is in the interior of the space of finite-surface-tension slopes, then there exists a minimal energy gradient phase μ_u of slope u .

Using a new geometric technique called *cluster swapping* (a variant of the Swendsen-Wang update for Fortuin-Kasteleyn clusters), we show that μ_u is unique if at least one of the following holds: $E = \mathbb{R}$, $d \in \{1, 2\}$, there exists a *rough gradient phase* of slope u , or u is irrational. When $d = 2$ and $E = \mathbb{Z}$, we show that the slopes of all *smooth phases* (a.k.a. *crystal facets*) lie in the dual lattice of $\mathcal{L}$.

In the case $E = \mathbb{Z}$ and $d = 2$, our results resolve and greatly generalize a number of conjectures of Cohn, Elkies, and Propp – one of which is that there is a unique ergodic Gibbs measure on domino tilings for each non-extremal slope. We also prove several theorems cited by Kenyon, Okounkov, and Sheffield in their recent exact solution of the dimer model on general planar lattices. In the case $E = \mathbb{R}$, our results generalize and extend many of the results in the literature on Ginzburg-Landau $\nabla\phi$ -interface models.

Résumé (Surfaces aléatoires). — Nous étudions les propriétés à grande échelle de “surfaces aléatoires” qui sont ici des applications aléatoires de $\mathbb{Z}^d$ (ou de grandes parties de $\mathbb{Z}^d$) à valeurs dans un ensemble E qui est égal à $\mathbb{R}$ ou $\mathbb{Z}$. Leur loi est donnée via un potentiel Gibbsien qui est une fonction convexe des gradients discrets (locaux) de la fonction, et qui est supposé invariant par rapport aux translations d’un véritable sous-réseau $\mathcal{L}$ de $\mathbb{Z}^d$. Ceci inclut beaucoup de modèles dits de hauteurs (pavages par dominos, glace carrée, le cristal harmonique, le modèle de Ginzburg-Landau $\nabla\phi$, le modèle SOS linéaire).

Nous établissons un principe variationnel qui caractérise les mesures de Gibbs d’une pente donnée comme minimiseurs de l’énergie libre spécifique, et un principe de grande déviations pour la mesure empirique pour les surfaces aléatoires sur des approximations par un réseau de domaines bornés. Nous montrons également que la tension de surface est strictement convexe et que lorsque la pente u définit une tension de surface finie, alors il existe une unique mesure de Gibbs μ_u de pente u ergodique par rapport à $\mathbb{L}$ et d’énergie minimale.

En utilisant une nouvelle idée géométrique de changement de clusters (qui est une variante de l’algorithme de Swendsen-Wang pour les modèles de percolation de Fortuin et Kasteleyn), nous montrons que la mesure μ_u est unique dès lors que l’une des conditions suivantes est vérifiée : $E = \mathbb{R}$, $d = \{1, 2\}$, il existe une mesure de Gibbs “irrégulière” de pente u , ou u est irrationnel.

Lorsque $d = 2$ et $E = \mathbb{Z}$, nous montrons que les pentes de toute mesure lisse (les faces du cristal) sont dans le réseau dual à $\mathcal{L}$. Lorsque $d = 2$ et $E = \mathbb{Z}$, nos résultats résolvent des conjectures de Cohn, Elkies et Propp ; nous montrons par exemple qu’il existe une seule mesure de Gibbs ergodique sur les pavages par dominos pour chaque pente non-extrémale. Nous établissons aussi des résultats utilisés par Kenyon, Okounkov et Sheffield pour résoudre de manière exacte le modèle de dimères sur des réseaux plans généraux. Lorsque $E = \mathbb{R}$, nos résultats généralisent nombre de résultats sur les modèles d’interfaces $\nabla\phi$ de Ginzburg-Landau.

CONTENTS

1. Introduction	1
1.1. Random surfaces and gradient Gibbs measures	2
1.2. Overview of remaining chapters	7
1.3. Two important special cases	11
2. Specific free energy and variational principle	15
2.1. Relative entropy review	15
2.2. Free energy	18
2.3. Specific free energy: existence via superadditivity	19
2.4. Specific free energy level set compactness	23
2.5. Minimizers of specific free energy are Gibbs measures	24
3. Ergodic/extremal decompositions and SFE	27
3.1. Funaki-Spohn gradient measures	27
3.2. Extremal and ergodic decompositions	30
3.3. Decompositions and SFE	32
4. Surface tension and energy	35
4.1. Energy bounds for ISAPs when $E = \mathbb{R}$	35
4.2. Torus approximations	42
4.3. Distance functions and interpolations	44
4.4. Gradient phase existence	48
4.5. A word on Lipschitz potentials	48
4.6. Examples: layered surfaces, non-intersecting lattice paths	49
5. Analytical results for Sobolev spaces	51
5.1. Orlicz-Sobolev spaces	52
5.2. Connection to the discrete settings	57
5.3. Low energy interpolations from L^1 and $L^{1,A}$ bounds	61
5.4. Approximation by “mostly linear” functions	67

6. Limit equalities and the variational principle	69
6.1. Limit equalities: $PBL(\mu)$, $FBL(\mu)$, and $SFE(\mu)$	69
6.2. Limit equalities in other settings	79
6.3. Variational principle	80
6.4. Variational principle in other settings	82
7. LDP for empirical measure profiles	83
7.1. Empirical measure profiles and statement of LDP	83
7.2. Proof of LDP	84
7.3. LDP in other settings	88
8. Cluster swapping	91
8.1. Introduction to cluster swapping	92
8.2. Monotonicity and log concavity	99
8.3. Measures on triplets	104
8.4. Height offset variables	106
8.5. Infinite cluster classifications	108
8.6. Gradient phase uniqueness and σ strict convexity	112
8.7. Height offset spectra	120
8.8. Example of slope- u gradient phase multiplicity	125
9. Discrete, two-dimensional gradient phases	129
9.1. Height offsets and main result	129
9.2. FKG inequality	130
9.3. Reduction to statement about $\{0, 1\}^{\mathbb{Z}^2}$ measures	131
9.4. Proof of statement about $\{0, 1\}^{\mathbb{Z}^2}$ measures	135
9.5. Ergodic gradient Gibbs measures of slope $u \in \partial U_\Phi$	149
10. Open Problems	151
10.1. Universality when $m = 1$ and $d = 2$	151
10.2. Universality when $m = 1$ and $d \geq 3$	152
10.3. Refinements of results proved here	152
10.4. Substantially different potentials	153
A. SFE and the lexicographic past	155
B. Summary of notations	161
Bibliography	169

CHAPTER 1

INTRODUCTION

The following is a fundamental problem of variational calculus: given a bounded open subset D of $\mathbb{R}^d$ and a free energy function $\sigma : D \times \mathbb{R}^d \times \mathbb{R}^{d \times m} \mapsto \mathbb{R}$, find the differentiable function $f : D \mapsto \mathbb{R}^m$ that (possibly subject to boundary conditions) minimizes the free energy integral:

$$\int_D \sigma(x, f(x), \nabla f(x)) \, dx.$$

Since the seventeenth century, these free-energy-minimizing functions have been popular models for determining (among other things) the shapes assumed by solid objects in the presence of outside forces: ropes suspended between poles, elastic sheets stretched to boundary conditions, and twisted or otherwise strained three-dimensional solids. They are also useful in modeling surfaces of water droplets and other phase interfaces. Rigorous formulations and solutions to these problems rank among the great achievements of classical analysis (including work by Fermat, Newton, Leibniz, the Bernoullis, Euler, Lagrange, Legendre, Jacobi, Hamilton, Weierstrass, etc. [49]) and play important roles in physics and engineering.

All of these models assume that matter is continuous and distributes force in a continuous way. One of the goals of statistical physics has become not merely to solve variational problems but to understand and, in some sense, to *justify* them in light of the fact that matter is comprised of individual, randomly behaving atoms. To this end, one begins by postulating a simple form for the local particle interactions: one approach – the one we will study in this work – is to represent the “atoms” of the solid crystal by points in a subset Λ of $\mathbb{Z}^d$, each of which has a “spatial position” given by a function $\phi : \Lambda \mapsto \mathbb{R}^m$, and to specify the interaction between the particles by a *Gibbs potential* Φ that possesses certain natural symmetries. The next step is to show that – at least in some “thermodynamic limit” – a random Gibbs configuration will approximate a free-energy-minimizing function like the ones described above.

Another problem, which has no analog in the deterministic, non-atomic classical theory, is the investigation of local statistics of a physical system. How likely are

particular microscopic configurations of atoms to occur as sub-configurations of a larger system? How are these occurrences distributed? To what extent is matter *homogenous* throughout small but non-microscopic regions? Our solutions to these problems will involve *large deviations principles*, which we precisely define later on.

Finally, we want to investigate more directly the connections between the Gibbs potential Φ and the kinds of behavior that can occur in these small but non-microscopic regions. This will require us to ask, given Φ , what are the “gradient phases” (i.e., the ergodic gradient Gibbs measures with finite specific energy) μ of a given slope? Does the μ -variance of the height difference of points n units apart remain bounded independently of n or does it tend to infinity with n ? When is the *surface tension* function σ (defined precisely in Chapter 4) strictly convex?

Before we state our results precisely and describe some of the previous work in this area, we will need several definitions. While we attempt to make our exposition relatively self-contained – and define the terms we use precisely – we will also draw heavily from the results in some standard texts: *Sobolev Spaces* by Adams [1] and recent extensions by Cianchi ([14], [15], [16]); *Large Deviations Techniques and Applications* by Dembo and Zeitouni [21]; *Large Deviations* by Deuschel and Stroock [25]; and *Gibbs Measures and Phase Transitions* by Georgii [42]. We will carefully state, if not prove, the outside theorems we use.

1.1. Random surfaces and gradient Gibbs measures

1.1.1. Gradient potentials. — The study of random functions ϕ from the lattice $\mathbb{Z}^d$ to a measure space $(E, \mathcal{E})$ is a central component of ergodic theory and statistical physics. In many classical models from physics (e.g., the Ising model, the Potts model, Shlosman’s plane rotor model), E is a space with a finite underlying measure λ , $\mathcal{E}$ is the Borel σ -field of E , and $\phi(x)$ has a physical interpretation as the *spin* (or some other internal property) of a particle at location x in a crystal lattice. (See e.g., [42].) In the models of interest to us, $(E, \mathcal{E})$ is a space with an infinite underlying measure λ – either $\mathbb{R}^m$ with Lebesgue measure or $\mathbb{Z}^m$ with counting measure – where $\mathcal{E}$ is the Borel σ -algebra of E and $\phi(x)$ usually has a physical interpretation as the *spatial position* of a particle (or the vertical height of a phase interface) at location x in a lattice. For example, if $m = d = 3$, ϕ could describe the spatial positions of the components of an elastic crystal; if $m = 1$ and $d = 2$, ϕ could describe the solid-on-solid or Ginzburg-Landau approximations of a phase interface [38].

Throughout the exposition, we denote by Ω the set of functions from $\mathbb{Z}^d$ to E and by $\mathcal{F}$ the Borel σ -algebra of the product topology on Ω . If $\Lambda \subset \mathbb{Z}^d$, we denote by $\mathcal{F}_\Lambda$ the smallest σ -algebra with respect to which $\phi(x)$ is measurable for all $x \in \Lambda$. We write $\mathcal{I}_\Lambda = \mathcal{F}_{\mathbb{Z}^d - \Lambda}$. We write $\Lambda \subset\subset \mathbb{Z}^d$ if Λ is a finite subset of $\mathbb{Z}^d$. A subset of Ω is called a *cylinder set* if it belongs to $\mathcal{F}_\Lambda$ for some $\Lambda \subset\subset \mathbb{Z}^d$. Let $\mathcal{F}$ be the smallest

σ -algebra on Ω containing the cylinder sets. We write $\mathcal{T}$ for the intersection of $\mathcal{T}_\Lambda$ over all finite subsets Λ of $\mathbb{Z}^d$; the sets in $\mathcal{T}$ are called *tail-measurable sets*.

We will also always assume that we are given a family Φ of measurable *potential functions* $\Phi_\Lambda : \Omega \mapsto \mathbb{R} \cup \{\infty\}$ (one for each finite subset Λ of $\mathbb{Z}^d$); each Φ_Λ is $\mathcal{F}_\Lambda$ measurable. We will further assume that Φ is invariant under the group Θ of translations of $\mathbb{Z}^d$ by members of some rank- d lattice $\mathcal{L}$ – i.e., if $s \in \mathcal{L}$, then $\Phi_{\Lambda+s}(\phi_s) = \Phi_\Lambda(\phi)$, where ϕ_s is defined by $\phi_s(i) = \phi(i-s)$. (In many applications, we can take $\mathcal{L} = \mathbb{Z}^d$.) We also assume that Φ is invariant under a group τ of measure-preserving translations of E – i.e., $\Phi_\Lambda(\phi) = \Phi_\Lambda(\tau\phi)$, where $\tau\phi$ is simply defined by $(\tau\phi)(x) = \tau(\phi(x))$. Potentials Φ satisfying the above requirements are called $\Theta \times \tau$ -invariant potentials or $\mathcal{L} \times \tau$ -invariant potentials. For all of our main results, we will assume that τ is the full group of translations of $\mathbb{Z}^m$ or $\mathbb{R}^m$; in this case, each $\Phi_\Lambda(\phi)$ is a function of the *gradient* of ϕ , written $\nabla\phi$ and defined by

$$\nabla\phi(x) = (\phi(x+e_1) - \phi(x), \phi(x+e_2) - \phi(x), \dots, \phi(x+e_d) - \phi(x)),$$

where e_i are the standard basis vectors of $\mathbb{Z}^d$. In this setting, we will refer to $\mathcal{L} \times \tau$ -invariant potentials as $\mathcal{L}$ -periodic or $\mathcal{L}$ -invariant gradient potentials. We use the term *shift-invariant* to mean $\mathcal{L}$ -invariant when $\mathcal{L} = \mathbb{Z}^d$. In some of our applications, we also restrict our attention to *nearest-neighbor potentials*, i.e., those potentials Φ for which $\Phi_\Lambda = 0$ unless Λ is a single pair of adjacent vertices in $\mathbb{Z}^d$. We say that Φ has *finite range* if there exists an r such that $\Phi_\Lambda = 0$ whenever the diameter of Λ is greater than r . For each finite subset Λ of $\mathbb{Z}^d$ we also define a *Hamiltonian*: $H_\Lambda(\phi) = \sum_{\Delta \cup \Lambda \neq \emptyset} \Phi_\Delta(\phi)$, where the sum is taken over finite subsets Δ of $\mathbb{Z}^d$.

We define the *interior Hamiltonian* of Λ , written $H_\Lambda^\circ(\phi)$, to be:

$$H_\Lambda^\circ(\phi) = \sum_{\Delta \subset \Lambda} \Phi_\Delta(\phi).$$

This is different from H_Λ because the former sum includes sets Δ that intersect Λ but are not strictly contained in Λ . On the other hand, H_Λ° is $\mathcal{F}_\Lambda^\tau$ measurable, which is not true of H_Λ . (This H_Λ° is sometimes called the *free boundary* Hamiltonian for Λ .)

1.1.2. Gibbs Measures. — To define Gibbs measures and gradient Gibbs measures, we will need some additional notation. Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be general measure spaces. A function $\pi : \mathcal{X} \times Y \mapsto [0, \infty]$ is called a *probability kernel* from $(Y, \mathcal{Y})$ to $(X, \mathcal{X})$ if

1. $\pi(\cdot|y)$ is a probability measure on $(X, \mathcal{X})$ for each fixed $y \in Y$, and
2. $\pi(A|\cdot)$ is $\mathcal{Y}$ -measurable for each fixed $A \in \mathcal{X}$.

Since a probability kernel maps each point in Y to a probability measure on X , we may interpret a probability kernel as giving the law for a random transition from