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Laurent Berger

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Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

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LUBIN–TATE GENERALIZATIONS OF THE p-ADIC FOURIER TRANSFORM

by Laurent Berger

Abstract. — Fresnel and de Mathan proved that the p-adic Fourier transform is surjective. We reinterpret their result in terms of analytic boundaries, and extend it beyond the cyclotomic case. We also give some applications of their result to Schneider and Teitelbaum's p-adic Fourier theory, in particular to generalized Mahler expansions and to the geometry of the character variety.

Résumé (*Généralisations Lubin-Tate de la transformation de Fourier p-adique*). — Fresnel et de Mathan ont montré que la transformation de Fourier p-adique est surjective. Nous réinterprétons leur résultat en termes de frontières analytiques, puis nous l'étendons au delà du cas cyclotomique. Nous donnons aussi des applications de leur résultat à la théorie de Fourier p-adique de Schneider et Teitelbaum, en particulier aux développements de Mahler généralisés et à la géométrie de la variété des caractères.

Introduction

The p-adic Fourier transform. — Let $\mathbf{C}_p$ be the completion of an algebraic closure of $\mathbf{Q}_p$, and let $\Gamma = \{\gamma \in \mathbf{C}_p \text{ such that } \gamma^{p^n} = 1 \text{ for some } n \geqslant 0\}$ be the set of roots of unity of p-power order. Let $c^0(\Gamma, \mathbf{C}_p)$ be the set of

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Laurent Berger, UMPA de l'ENS de Lyon, UMR 5669 du CNRS • *E-mail :* `laurent.berger@ens-lyon.fr` • *Url :* `https://perso.ens-lyon.fr/laurent.berger/`

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sequences $\{z_\gamma\}_{\gamma\in\Gamma}$ with $z_\gamma \in \mathbf{C}_p$ and such that $z_\gamma \to 0$ (namely for every $\varepsilon > 0$, the set of γ such that $|z_\gamma| \geqslant \varepsilon$ is finite), and let $C^0(\mathbf{Z}_p, \mathbf{C}_p)$ be the space of continuous functions $\mathbf{Z}_p \to \mathbf{C}_p$. For every $\gamma \in \Gamma$, the function $a \mapsto \gamma^a$ belongs to $C^0(\mathbf{Z}_p, \mathbf{C}_p)$.

Definition. — *The* Fourier transform *of $z \in c^0(\Gamma, \mathbf{C}_p)$ is the function $\mathcal{F}(z) : \mathbf{Z}_p \to \mathbf{C}_p$ given by $a \mapsto \sum_{\gamma\in\Gamma} z_\gamma \cdot \gamma^a$.*

Fresnel and de Mathan proved (see [7, 8, 9] as well as [12] for a different proof) the following result:

Theorem. — *The Fourier transform $\mathcal{F} : c^0(\Gamma, \mathbf{C}_p) \to C^0(\mathbf{Z}_p, \mathbf{C}_p)$ is surjective, and moreover, $\mathcal{F} : c^0(\Gamma, \mathbf{C}_p)/\ker \mathcal{F} \to C^0(\mathbf{Z}_p, \mathbf{C}_p)$ is an isometry.*

Because of the appearance of roots of unity, the p-adic Fourier transform can be seen as a cyclotomic construction. In this paper, we generalize the definition of the Fourier transform as well as Fresnel and de Mathan's theorem beyond the cyclotomic case. We then give a mostly independent application of their theorem to Schneider and Teitelbaum's p-adic Fourier theory [14].

Analytic boundaries. — For the first generalization, consider the dual of the p-adic Fourier transform. The dual of $c^0(\Gamma, \mathbf{C}_p)$ is $\ell^\infty(\Gamma, \mathbf{C}_p)$, the set of bounded sequences. The dual of $C^0(\mathbf{Z}_p, \mathbf{C}_p)$ is isomorphic to $\mathcal{E}^+_{\mathbf{C}_p} = \mathbf{C}_p \otimes_{\mathcal{O}_{\mathbf{C}_p}} \mathcal{O}_{\mathbf{C}_p}[\![X]\!]$ (via the Amice transform that sends a measure μ to $\mathcal{A}_\mu(X) = \sum_{n\geqslant 0} \mu(a \mapsto \binom{a}{n}) \cdot X^n)$.

The dual of the Fourier transform is hence a map $\mathcal{F}' : \mathcal{E}^+_{\mathbf{C}_p} \to \ell^\infty(\Gamma, \mathbf{C}_p)$. It is easy to see that this map is given by $f(X) \mapsto \{f(\gamma-1)\}_{\gamma\in\Gamma}$. Fresnel and de Mathan's theorem is then equivalent to the claim that $\mathcal{F}'$ is an isometry on its image, namely that $\|f\|_D = \sup_{\gamma\in\Gamma} |f(\gamma-1)|$, where $D = \mathfrak{m}_{\mathbf{C}_p}$ is the p-adic open unit disk.

Definition. — *A subset $A = \{a_n\}_{n\geqslant 1} \subset D$ is an* analytic boundary *if $|a_n| \to 1$ as $n \to +\infty$ and if for every $f \in \mathcal{E}^+_{\mathbf{C}_p}$ we have $\|f\|_D = \|f\|_A := \sup_{n\geqslant 1} |f(a_n)|$.*

Fresnel and de Mathan's theorem is then equivalent to the claim that $\{\gamma-1, \gamma \in \Gamma\}$ is an analytic boundary. We prove that the same holds if A is the set of torsion points of a Lubin–Tate formal group attached to a finite extension of $\mathbf{Q}_p$ and, even more generally, if A is the set of iterated roots of a certain class of power series, which we call Lubin–Tate-like (LT-like) power series. Let q be a power of p.

Definition. — *An* LT-like power series *(of Weierstrass deg q) is a power series $P(X) = \sum_{n\geqslant 1} p_n X^n \in \mathcal{O}_{\mathbf{C}_p}[\![X]\!]$ with $0 < \mathrm{val}_p(p_1) \leqslant 1$, $p_q \in \mathcal{O}^\times_{\mathbf{C}_p}$, and $P(X) \equiv p_q X^q \bmod p_1$.*

If $P(X)$ is as above, let $\Lambda(P) = \{z \in D$ such that $P^{\circ n}(z) = 0$ for some $n \geqslant 0\}$. The following result is theorem 1.2.2:

Theorem A. — *If P is LT like, then $\Lambda(P)$ is an analytic boundary.*

If $P(X) = (1+X)^p - 1$, then $\Lambda(P) = \{\gamma - 1, \gamma \in \Gamma\}$, and theorem A implies the result of Fresnel and de Mathan. The proof of theorem A is very similar to Fresnel and de Mathan's proof of their result.

***p*-adic Fourier theory**. — For the second generalization, let F be a finite extension of $\mathbf{Q}_p$ of degree d, with ring of integers $\mathcal{O}_F$. Let X_{tor} denote the set of finite order characters $(\mathcal{O}_F, +) \to (\mathbf{C}_p^\times, \times)$. Given $z \in c^0(X_{\mathrm{tor}}, \mathbf{C}_p)$, its Fourier transform is the function $\mathcal{F}(z) : \mathcal{O}_F \to \mathbf{C}_p$ defined by $a \mapsto \sum_{g \in X_{\mathrm{tor}}} z_g \cdot g(a)$. It is easy to see (theorem 2.1.1) that Fresnel and de Mathan's theorem implies that $\mathcal{F} : c^0(X_{\mathrm{tor}}, \mathbf{C}_p) \to C^0(\mathcal{O}_F, \mathbf{C}_p)$ is surjective. We give an application of this observation to p-adic Fourier theory.

Let e be the ramification index of F, let π be a uniformizer of $\mathcal{O}_F$, and let $q = \operatorname{card} \mathcal{O}_F/\pi$. Let LT be the Lubin–Tate formal $\mathcal{O}_F$-module attached to π, let X be a coordinate on LT, and let $\log_{\mathrm{LT}}(X)$ be the logarithm of LT. For $n \geqslant 0$, let $P_n(Y) \in F[Y]$ be the polynomial defined by $\exp(Y \cdot \log_{\mathrm{LT}}(X)) = \sum_{n \geqslant 0} P_n(Y) X^n$.

When $F = \mathbf{Q}_p$ and $\mathrm{LT} = \mathbf{G}_{\mathrm{m}}$, we have $P_n(Y) = \binom{Y}{n}$. The family $\{\binom{Y}{n}\}_{n \geqslant 0}$ forms a Mahler basis of $\mathbf{Z}_p$. In addition, by a theorem of Amice [1], every locally analytic function $\mathbf{Z}_p \to \mathbf{C}_p$ can be written as $x \mapsto \sum_{n \geqslant 0} c_n \binom{x}{n}$, where $\{c_n\}_{n \geqslant 0}$ is a sequence of $\mathbf{C}_p$ such that there exists $r > 1$ satisfying $|c_n| \cdot r^n \to 0$.

In their work [14] on p-adic Fourier theory, Schneider and Teitelbaum generalized this last result to $F \neq \mathbf{Q}_p$. They proved the existence of an element $\Omega \in \mathcal{O}_{\mathbf{C}_p}$, with $\mathrm{val}_p(\Omega) = 1/(p-1) - 1/e(q-1)$, such that $P_n(a\Omega) \in \mathcal{O}_{\mathbf{C}_p}$ for all $a \in \mathcal{O}_F$. The power series $G(X) = \exp(\Omega \cdot \log_{\mathrm{LT}}(X)) - 1$ therefore belongs to $\mathrm{Hom}_{\mathcal{O}_{\mathbf{C}_p}}(\mathrm{LT}, \mathbf{G}_{\mathrm{m}})$. One of the main results of p-adic Fourier theory is the following (prop 4.5 and theo 4.7 of [14]):

Theorem. — *If $\{c_m\}_{m \geqslant 0}$ is a sequence of $\mathbf{C}_p$ such that there exists $r > 1$ satisfying $|c_m| \cdot r^m \to 0$, then $a \mapsto \sum_{m \geqslant 0} c_m P_m(a\Omega)$ is a locally F-analytic function $\mathcal{O}_F \to \mathbf{C}_p$.*

Conversely, every locally F-analytic function $\mathcal{O}_F \to \mathbf{C}_p$ has a unique such expansion.

If we only ask that $c_m \to 0$, then $a \mapsto \sum_{m \geqslant 0} c_m P_m(a\Omega)$ is a continuous function $\mathcal{O}_F \to \mathbf{C}_p$. We therefore get a map $c^0(\mathbf{N}, \mathbf{C}_p) \to C^0(\mathcal{O}_F, \mathbf{C}_p)$, whose image contains all locally F-analytic functions. If $F = \mathbf{Q}_p$, this map is an isomorphism. In general, Fresnel and de Mathan's theorem and some computations in p-adic Fourier theory imply that the map is surjective and is noninjective if $F \neq \mathbf{Q}_p$. Using the fact that every element of $C^0(\mathbf{Z}_p, \mathbf{C}_p)$ can be written in one and only one way as $x \mapsto \sum_{n \geqslant 0} \lambda_n \binom{x}{n}$, where $\lambda \in c^0(\mathbf{N}, \mathbf{C}_p)$, we reformulate this result using the following definition:

DEFINITION. — *The* Peano map $T : C^0(\mathbf{Z}_p, \mathbf{C}_p) \to C^0(\mathcal{O}_F, \mathbf{C}_p)$ *is the map given by*

$$T : \Big[x \mapsto \sum_{n \geqslant 0} \lambda_n \binom{x}{n}\Big] \mapsto \Big[a \mapsto \sum_{n \geqslant 0} \lambda_n P_n(a\Omega)\Big].$$

THEOREM B. — *The Peano map $T : C^0(\mathbf{Z}_p, \mathbf{C}_p) \to C^0(\mathcal{O}_F, \mathbf{C}_p)$ is surjective and is noninjective if $F \neq \mathbf{Q}_p$.*

This is corollary 2.2.1. By Schneider and Teitelbaum's theorem recalled above, $T : C^{\mathrm{la}}(\mathbf{Z}_p, \mathbf{C}_p) \to C^{F\text{-}\mathrm{la}}(\mathcal{O}_F, \mathbf{C}_p)$ is an isomorphism. So one can think of T as some Peano-like map: a surjective noninjective limit of isomorphisms, from a 1-dimensional object to a d-dimensional object.

The character variety. — The rigid analytic p-adic open unit disk $\mathfrak{B}$ is a parameter space for characters $(\mathbf{Z}_p, +) \to (\mathbf{C}_p^\times, \times)$: if K is a closed subfield of $\mathbf{C}_p$, a point $z \in \mathfrak{B}(K)$ corresponds to the character $\eta_z : a \mapsto (1+z)^a$ and all K-valued continuous characters are of this form. In particular, all continuous characters are locally analytic.

If F is a finite extension of $\mathbf{Q}_p$ of degree d, then $\mathcal{O}_F \simeq \mathbf{Z}_p^d$ and $\mathfrak{B}^d$ is then a parameter space for characters $(\mathcal{O}_F, +) \to (\mathbf{C}_p^\times, \times)$. Schneider and Teitelbaum constructed in [14] a 1-dimensional rigid analytic group variety $\mathfrak{X} \subset \mathfrak{B}^d$ over F called the character variety, whose closed points in an extension K/F parameterize locally F-analytic characters $\mathcal{O}_F \to K^\times$. They showed that over $\mathbf{C}_p$, the variety $\mathfrak{X}$ becomes isomorphic to $\mathfrak{B}$.

Let $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{B}^d)$ denote the ring of bounded functions on $\mathfrak{B}^d$ defined over $\mathbf{C}_p$ and likewise for $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X})$. We have $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X}) \simeq \mathcal{E}^+_{\mathbf{C}_p}$, and $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{B}^d)$ is likewise isomorphic to the ring of bounded functions in d variables. The restriction-to-$\mathfrak{X}$ map $\mathrm{res}_{\mathfrak{X}} : \mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{B}^d) \to \mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X})$ is injective by [4]. By p-adic Fourier theory, $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X})$ is the dual of $C^0(\mathbf{Z}_p, \mathbf{C}_p)$, $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{B}^d)$ is the dual of $C^0(\mathcal{O}_F, \mathbf{C}_p)$, and $\mathrm{res}_{\mathfrak{X}}$ is the dual of the Peano map T.

Theorem B now implies the following result (theorem 2.3.1):

THEOREM C. — *The map* $\mathrm{res}_{\mathfrak{X}} : \mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{B}^d) \to \mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X})$ *is an isometry on its image.*

In the isomorphism between $\mathfrak{X}$ and $\mathfrak{B}$, we have $\mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X}) \simeq \mathcal{E}^+_{\mathbf{C}_p}$, and the set X_{tor} of torsion characters $(\mathcal{O}_F, +) \to (\mathbf{C}_p^\times, \times)$ corresponds to $\mathrm{LT}[\pi^\infty]$. Theorem A applied to $P(X) = [\pi](X)$ implies the following result (theorem 2.3.3):

THEOREM D. — *If $f \in \mathcal{O}^b_{\mathbf{C}_p}(\mathfrak{X})$, then $\|f\|_{\mathfrak{X}} = \sup_{\kappa \in X_{\mathrm{tor}}} |f(\kappa)|$.*

Theorem A is proved in §1, and theorems B, C, and D are proved in §2.

1. Construction of analytic boundaries

The goal of this section is to state and prove theorem A.

1.1. p-adic holomorphic functions and analytic boundaries. — We recall some standard facts about holomorphic functions on the p-adic open unit disk (see [11] or [13]) and define analytic boundaries. Let $D = \mathfrak{m}_{\mathbf{C}_p}$ be the p-adic open unit disk. Let $\mathcal{E}^+_{\mathbf{C}_p} = \mathbf{C}_p \otimes_{\mathcal{O}_{\mathbf{C}_p}} \mathcal{O}_{\mathbf{C}_p}[\![X]\!]$ be the ring of bounded holomorphic functions on D, and let $\mathcal{R}^+_{\mathbf{C}_p}$ be the ring of holomorphic functions on D. If $f(X) = \sum_{n\geqslant 0} f_n Z^n \in \mathcal{R}^+_{\mathbf{C}_p}$ and $\mu > 0$, we let $V(f,\mu) = \inf_{n\geqslant 0} \mathrm{val}_p(f_n) + \mu n$. If $\mu \in \mathbf{Q}_{>0}$, then $V(f,\mu) = \inf_{z\in D, \mathrm{val}_p(z)=\mu} \mathrm{val}_p(f(z))$. The function $\mu \mapsto V(f,\mu)$ is continuous, increasing, and piecewise affine. We have $V(fg,\mu) = V(f,\mu) + V(g,\mu)$. If $f \in \mathcal{E}^+_{\mathbf{C}_p}$, then $V(f,0)$ is also defined, and $V(f,0) = -\log_p \|f\|_D$. We say that $\mu > 0$ is a critical valuation if there exists $i \neq j$ such that $V(f,\mu) = \mathrm{val}_p(f_i) + \mu i = \mathrm{val}_p(f_j) + \mu j$. Recall that f has a zero of valuation μ if and only if μ is a critical valuation, and that the critical valuations of f, as well as the number of zeroes of f having that valuation, can be read on the Newton polygon of f.

Divisors are defined in §4 of [11]. In this paper, we only consider divisors that are an infinite formal product $\prod_{k\geqslant 1} D_k(X)$, where for each k, $D_k(X)$ is a polynomial such that $D_k(0) = 1$ and all the roots of D_k are of valuation μ_k, where $\{\mu_k\}_{k\geqslant 1}$ is a strictly decreasing sequence converging to 0. We then have $V(D_k,\mu) = 0$ if $\mu \geqslant \mu_k$ and $V(D_k,\mu) = \deg D_k \cdot (\mu - \mu_k)$ if $\mu \leqslant \mu_k$.

Proposition 1.1.1. — *Let $\prod_{k\geqslant 1} D_k(X)$ be a divisor and take $\eta > 0$.*

There exists $f(X) \in \mathcal{R}^+_{\mathbf{C}_p}$ such that $f(0) = 1$; f is divisible by D_k for all $k \geqslant 1$; and for all $\mu > 0$, we have $\sum_{k\geqslant 1} V(D_k,\mu) \geqslant V(f,\mu) \geqslant \sum_{k\geqslant 1} V(D_k,\mu) - \eta$.

Proof. — This is theorem 1 of [7]. See theorem 25.5 of [6] for a full proof, noting that $A_b(d(0,r^-))$ should be $A(d(0,r^-))$ in the statement of ibid. □

We now define analytic boundaries. Since D is a separable topological space, there are plenty of countable sets $A = \{a_n\}_{n\geqslant 1} \subset D$ such that $\|f\|_D = \|f\|_A := \sup_{n\geqslant 1} |f(a_n)|$ for all $f \in \mathcal{E}^+_{\mathbf{C}_p}$. We are interested in those sets A such that $|a_n| \to 1$ as $n \to +\infty$.

Definition 1.1.2. — We say that $A = \{a_n\}_{n\geqslant 1} \subset D$ is an analytic boundary if $|a_n| \to 1$ as $n \to +\infty$ and if for every $f \in \mathcal{E}^+_{\mathbf{C}_p}$ we have $\|f\|_D = \|f\|_A$.

Lemma 1.1.3. — *If A is an analytic boundary and $h \neq 0 \in \mathcal{E}^+_{\mathbf{C}_p}$, then $A' = A \setminus \{a \in A$ such that $h(a) = 0\}$ is also an analytic boundary.*