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CONTROLLED OBJECTS IN LEFT-EXACT ∞-CATEGORIES AND THE NOVIKOV CONJECTURE

by Ulrich Bunke, Denis-Charles Cisinski, Daniel Kasprowski & Christoph Winges

Abstract. — We associate to every G-bornological coarse space X and every left-exact ∞-category with G-action a left-exact infinity-category of equivariant X-controlled objects. Postcomposing with algebraic K-theory leads to new equivariant coarse homology theories. This allows us to apply the injectivity results for assembly maps by Bunke, Engel, Kasprowski and Winges to the algebraic K-theory of left-exact ∞-categories.

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Résumé (*Objets contrôlés dans les ∞-catégories exactes à gauche et la conjecture de Novikov*). — Nous associons à tout espace G-bornologique grossier X et à toute ∞-catégorie exacte à gauche munie d'une G-action une ∞-catégorie exacte à gauche des objets équivariants X-contrôlés. En considérer la K-théorie nous conduit à de nouvelles homologies grossières équivariantes. Cela nous permet d'appliquer les résultats d'injectivité pour les morphismes d'assemblage dûs à Bunke, Engel, Kasprowski et Winges à la K-théorie des ∞-catégorie exactes à gauche.

1. Introduction

This paper concerns the construction of G-equivariant coarse homology theories in the sense of [18, Def. 3.10]. Given a left-exact ∞-category with G-action, we first construct a functor which associates to every G-bornological coarse space a new left-exact ∞-category of equivariant controlled objects. The coarse homology theory is then obtained by composing this functor with a localising invariant from left-exact ∞-categories to some target stable ∞-category. We employ these equivariant coarse homology theories in order to study properties of assembly maps.

Any equivariant coarse homology theory can be restricted (see Example 7.1.4) to a functor, denoted by $M\colon G\mathbf{Orb} \to \mathbf{M}$ for the moment, on the orbit category $G\mathbf{Orb}$. We then consider the assembly map

$$\mathrm{Ass}_{\mathbf{Fin},M}\colon \operatorname*{colim}_{G_{\mathbf{Fin}}\mathbf{Orb}} M \to M(*) \ , \tag{1.0.1}$$

which approximates the value $M(*)$ of M on the final object of $G\mathbf{Orb}$ by its values on the subcategory $G_{\mathbf{Fin}}\mathbf{Orb}$ of orbits with finite stabilisers. The word *Novikov conjecture* from the title refers to the assertion that this assembly map is split injective under certain conditions. We describe the history of this term in greater detail in Remark 1.1.19.

The relevance of coarse homology theories for the verification of split injectivity of the assembly map (1.0.1) stems from the axiomatic approach to this question developed in [19], which builds on a long tradition of proofs using similar methods [24, 6, 42, 32]. The essential assumption on the functor M is the CP-condition, which we recall in Definition 7.1.3. It requires that M arises from an equivariant coarse homology theory, as in Example 7.1.4, and that this coarse homology theory has various additional properties.

We verify that the equivariant coarse homology theory constructed from a left-exact ∞-category with G-action $\mathbf{D}$ and algebraic K-theory in place of the localising invariant has the required properties to ensure that the resulting functor, denoted by $K\mathbf{D}_G\colon G\mathbf{Orb} \to \mathbf{M}$ in (1.1.4), is a CP-functor. This approach subsumes various previously known cases but also adds new examples of functors on the orbit category which are therefore known to satisfy the CP-condition.

In Section 1.1, we start with a more detailed discussion of the construction of functors on the orbit category from left-exact ∞-categories with G-action and localising invariants. In particular, we explain how some of the classical examples of functors on the orbit category can be considered as special cases of our general construction. In Theorem 1.1.6, we provide a sample split injectivity result for the assembly map derived by combining [19] with the results of the present paper.

In Section 1.2, we give a detailed overview of the construction of coarse homology theories from left-exact ∞-categories with G-action. The technical details of this construction account for the main body of the paper.

1.1. Split injectivity of assembly maps. — Let G be a group. The orbit category $G\mathbf{Orb}$ is the category of transitive G-sets and equivariant maps. For a family $\mathcal{F}$ of subgroups of G (Definition 7.3.10), let $G_{\mathcal{F}}\mathbf{Orb}$ denote the full subcategory of the orbit category consisting of G-sets with stabilisers in $\mathcal{F}$ (Definition 7.3.11).

We consider a functor $M\colon G\mathbf{Orb} \to \mathbf{M}$ with a cocomplete target ∞-category. For any pair of families $\mathcal{F}'$ and $\mathcal{F}$ of subgroups of G such that $\mathcal{F}' \subseteq \mathcal{F}$, we then have a relative assembly map (see Definition 7.3.16)

$$\mathrm{Ass}^{\mathcal{F}}_{\mathcal{F}',M}\colon \operatorname*{colim}_{G_{\mathcal{F}'}\mathbf{Orb}} M \to \operatorname*{colim}_{G_{\mathcal{F}}\mathbf{Orb}} M\ . \tag{1.1.1}$$

It is a morphism between objects of $\mathbf{M}$ and induced by the inclusion of the index categories of the colimits in (1.1.1).

A natural question about the assembly map is whether it is an equivalence or at least split injective. The split injectivity question has been studied axiomatically in [19]. In this approach, the main assumption on the functor M is that it is a CP-functor.

As said above, being a CP-functor requires that M extends to an equivariant coarse homology theory in a particular way, and that this equivariant coarse homology theory has various additional properties. Our main contribution in this direction is Theorem 1.1.5 below, stating that $K\mathbf{D}_G$ is a CP-functor. We start with a precise description of this functor.

A left-exact ∞-category is an ∞-category which contains a zero object and admits all finite limits. A functor between left exact ∞-categories is called left-exact if it preserves finite limits. We let $\mathbf{Cat}^{\mathrm{Lex}}_{\infty,*}$ denote the large ∞-category of small left-exact ∞-categories and left-exact functors, see Example 2.1.3.

Small left-exact ∞-categories with G-actions are objects of the functor category $\mathbf{Fun}(BG, \mathbf{Cat}^{\mathrm{Lex}}_{\infty,*})$. For the following, we fix a small left-exact ∞-category with G-action $\mathbf{D}$.

The group G considered as a G-set with the G-action by left translations is an object of $G\mathbf{Orb}$. Its group of automorphisms is G acting by right translations. By BG we denote the groupoid consisting of a single object with group of automorphisms G. Sending the unique object in BG to the free orbit G provides an embedding

$$j\colon BG \to G\mathbf{Orb}\ . \tag{1.1.2}$$

Since $\mathbf{Cat}^{\mathrm{Lex}}_{\infty,*}$ admits all small colimits (Proposition 2.1.32), we can form the left Kan extension

$$j_!\mathbf{D}\colon G\mathbf{Orb} \to \mathbf{Cat}^{\mathrm{Lex}}_{\infty,*} \tag{1.1.3}$$

of $\mathbf{D}$ along j. We further compose $j_!\mathbf{D}$ with the algebraic K-theory functor $K\colon \mathbf{Cat}^{\mathrm{Lex}}_{\infty,*} \to \mathbf{Sp}$ in order to define the functor

$$K\mathbf{D}_G := K \circ j_!\mathbf{D}\colon G\mathbf{Orb} \to \mathbf{Sp}\ , \tag{1.1.4}$$

see Definition 7.1.6 and Definition 7.2.1 for details.

We refer to Definition 7.1.9 for a precise definition of the term "hereditary CP-functor".

Theorem 1.1.5 (Corollary 7.2.7). — *The functor $K\mathbf{D}_G$ is a hereditary CP-functor.*

As said above, the CP-condition on $K\mathbf{D}_G$ allows us to apply the axiomatic approach to injectivity results for assembly maps developed in [19]. The following theorem describes a typical example of such an application:

Theorem 1.1.6. — *Assume that*

(1) *G admits a finite-dimensional CW-model for the classifying space $E_{\mathbf{Fin}}G$.*
(2) *G is a finitely generated subgroup of a linear group over a commutative ring with unit or of a virtually connected Lie group.*

Then the assembly map

$$\mathrm{Ass}^{\mathbf{All}}_{\mathbf{Fin},K\mathbf{D}_G}\colon \operatorname*{colim}_{G_{\mathbf{Fin}}\mathbf{Orb}} K\mathbf{D}_G \to \operatorname*{colim}_{G\mathbf{Orb}} K\mathbf{D}_G$$

admits a left inverse.

Using that $*$ is the final object of $G\mathbf{Orb}$, we can identify the target of this assembly map with $K\mathbf{D}_G(*)$ in order to get the version (1.0.1). Theorem 1.1.5 exhibits Theorem 1.1.6 as a consequence of Theorem 7.3.21. For a detailed review of the general results of [19] involving more complicated assumptions on G (e.g. the condition of finite decomposition complexity), we refer to Section 7.3.

The assumptions required in the theorems listed in Section 7.3 can be separated into assumptions on the group G and the families $\mathcal{F}', \mathcal{F}$ on the one hand, and the assumption on the functor M being a CP-functor, see Definition 7.1.3, on the other. The present paper contributes to the latter. In particular, we make no attempt to enlarge the class of groups for which injectivity results are known.

The Farrell–Jones conjecture predicts that the assembly map $\mathrm{Ass}^{\mathbf{All}}_{\mathbf{Vcyc},K\mathbf{D}_G}$ for the family of virtually cyclic subgroups **Vcyc** is an equivalence. Generalising work of Bartels [7], we show in Theorem 7.3.18 that the relative assembly map $\mathrm{Ass}^{\mathbf{Vcyc}}_{\mathbf{Fin},K\mathbf{D}_G}$ is always split injective. Therefore, Theorem 1.1.6 can also be read as providing evidence towards the Farrell–Jones conjecture. In fact, the coarse homology theories constructed in the present paper are used crucially in [21] in order to extend proofs of the Farrell–Jones conjecture from the linear case (see Example 1.1.7) to the version stated above.

We now explain the relation between the functor (1.1.4) and examples of functors whose assembly maps have been classically considered. We start by recalling their constructions.

Example 1.1.7. — The motivating and guiding example for our approach is the equivariant algebraic K-theory functor

$$K\mathbf{A}_G\colon G\mathbf{Orb}\to\mathbf{Sp}$$

associated to an additive category $\mathbf{A}$ with a strict G-action. We refer to this case as the linear case as opposed to the derived case. The functor $K\mathbf{A}_G$ was first constructed in [26].

In the following, we give a quick alternative construction of $K\mathbf{A}_G$ which is analogous to the construction of the functor in (1.1.4) above. We consider the large ∞-category $\mathbf{Add}_\infty$ of small additive categories obtained from the category of small additive categories and additive functors by inverting equivalences. We then interpret $\mathbf{A}$ as an object $\mathbf{A}_\infty$ of $\mathbf{Fun}(BG,\mathbf{Add}_\infty)$. We denote the left Kan extension of $\mathbf{A}_\infty$ along j by

$$j_!\mathbf{A}_\infty\colon G\mathbf{Orb}\to\mathbf{Add}_\infty\ .$$

Finally, we let

$$K^{\mathrm{Add}}\colon \mathbf{Add}_\infty\to\mathbf{Sp} \tag{1.1.8}$$

be the nonconnective K-theory functor for additive categories (constructed by Pedersen–Weibel [40] and Schlichting [48]). We then define the composed functor

$$K\mathbf{A}_G:=K^{\mathrm{Add}}\circ j_!\mathbf{A}_\infty\colon G\mathbf{Orb}\to\mathbf{Sp} \tag{1.1.9}$$

(compare with Definition 7.1.6).

Theorem 1.1.10 ([19, Ex. 1.10], [19, Ex. 2.6]). — *The functor $K\mathbf{A}_G$ is a hereditary CP-functor.*

The argument for this result given in [19] is short but heavily uses results from [18] and the quite technical paper [20]. Theorem 1.1.10 allows us to apply the split injectivity results for assembly maps from [19] to $K\mathbf{A}_G$. ♦