Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 153
Fascicule 2

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 459-494

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 2, tome 153, juin 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (2), 2025, p. 459-494

UNIVERSAL INVARIANTS, THE CONWAY POLYNOMIAL AND THE CASSON–WALKER–LESCOP INVARIANT

by Adrien Casejuane & Jean-Baptiste Meilhan

Abstract. — We give a general surgery formula for the Casson–Walker–Lescop invariant of closed 3-manifolds, seen as the leading term of the LMO invariant, in a purely diagrammatic and combinatorial way. This provides a new viewpoint on a formula established by C. Lescop for her extension of the Walker invariant. A central ingredient in our proof is an explicit identification of the coefficients of the Conway polynomial as combinations of coefficients in the Kontsevich integral. This latter result relies on general "factorization formulas" for the Kontsevich integral coefficients.

Résumé (*Invariants universels, polynôme de Conway et invariant de Casson-Walker-Lescop*). — Nous donnons une formule de chirurgie pour l'invariant de Casson-Walker-Lescop des 3-variétés closes, vu comme le terme dominant de l'invariant LMO, par des méthodes purement diagrammatiques et combinatoires. Ceci fournit un point de vue nouveau sur une formule due à C. Lescop pour son extension de l'invariant de Walker. Un ingrédient central est une identification explicite des coefficients du polynôme de Conway comme combinaison de coefficients de l'intégrale de Kontsevich. Ce dernier résultat repose sur des «formules de factorisation» générales pour les coefficients de l'intégrale de Kontsevich.

Texte reçu le 24 novembre 2023, modifié le 16 juin 2024, accepté le 9 juillet 2024.

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Mathematical subject classification (2010). — 57M27, 57M25.

Key words and phrases. — Kontsevich integral, LMO invariant, Casson invariant, Alexander–Conway polynomial, Jacobi diagrams.

The second author is partially supported by the project AlMaRe (ANR-19-CE40-0001-01) of the ANR.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2025/459/$ 5.00

doi:10.24033/bsmf.2903

1. Introduction

A. Casson defined in 1985 an invariant of integral homology spheres by counting conjugacy classes of irreducible $SU(2)$–representations of the fundamental group [1, 5]. The Casson invariant was extended, first to rational homology spheres by K. Walker [19] and then to all oriented closed 3-manifolds by C. Lescop [12], via surgery formulas. We denote by λ_L this *Casson–Walker–Lescop invariant.*

In [10], T. T. Q. Le, J. Murakami and T. Ohtsuki defined an invariant of closed oriented 3-manifolds. This *LMO invariant* is built from the Kontsevich integral [8] of a surgery presentation, i.e. a framed link in S^3. The *Kontsevich integral* of a framed n-component link takes values in a graded space of chord diagrams on n circles, while the LMO invariant lives in a graded space of trivalent diagrams; the procedure for extracting the latter invariant from the former one relies on a family of sophisticated combinatorial maps ι_n that "replace circles by sums of trees". The Kontsevich integral is universal among $\mathbb{Q}$-valued Vassiliev invariants in the sense that any such invariant factors through the Kontsevich integral. Likewise, the LMO invariant is universal among $\mathbb{Q}$-valued finite type invariants of rational homology spheres. Both invariants admit purely combinatorial and diagrammatic definitions, although concrete computations are, in general, rather difficult.

A striking result is that the leading term of the LMO invariant, i.e. the coefficient of the lowest-degree trivalent diagram , is, up to a known factor, the Casson–Walker–Lescop invariant [9, 6]. This provides, in principle, a combinatorial procedure for computing the Casson–Walker–Lescop invariant from a surgery presentation by computing the Kontsevich integral and keeping track of the coefficients of chord diagrams that produce a diagram  under the LMO procedure. This paper shows how this can be done completely explicitly in terms of (classical) link invariants. Our first main result is as follows:

THEOREM 1 (Thm. 5.1). — *Let L be a framed oriented n-component link in S^3, and let $\mathbb{L}$ denote its linking matrix. Let S^3_L be the result of surgery on S^3 along L. The Casson–Walker–Lescop invariant $\lambda_L(S^3_L)$ is given by*

$$\frac{(-1)^{\sigma_-(L)}\det \mathbb{L}}{8}\sigma(L) + (-1)^{n+\sigma_-(L)}\sum_{k=1}^{n}\sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}}(-1)^{n-k}\det \mathbb{L}_{\check{I}}\mu_k(L_I),$$

where

- $\mathbb{L}_{\check{I}}$ *is the matrix obtained from* $\mathbb{L}$ *by deleting the lines and column indexed by a subset I of* $\{1,\dots,n\}$*;*
- $\sigma_+(L)$ *and* $\sigma_-(L)$ *denote, respectively, the number of positive and negative eigenvalues of* $\mathbb{L}$*, and* $\sigma(L) = \sigma_+(L) - \sigma_-(L)$*; and*
- μ_k *is a k-component framed link invariant which is explicitly determined by the coefficients of* $\mathbb{L}$ *and the Conway polynomial.*

We do not give here the general explicit formula for the invariant μ_k, which is postponed to Theorem 4.10. Let us only give here the formulas for the first two of these invariants: for a framed knot K, we have $\mu_1(K) = \frac{1}{24}fr(K)^2 - c_2(K) + \frac{1}{12}$, and if $L = K_1 \cup K_2$ is a 2-component framed link, then $\mu_2(L)$ is given by

$$\frac{1}{12}\mathrm{lk}(L)^3 + \frac{fr(K_1)+fr(K_2)}{12}\mathrm{lk}(L)^2 + \mathrm{lk}(L)\Big(c_2(K_1)+c_2(K_2)-\frac{1}{12}\Big) - c_3(L),$$

where c_k denotes the coefficient of z^k in the Conway polynomial. These two invariants are involved in the case $n = 2$ of Theorem 1, which recovers a result of S. Matveev and M. Polyak [13, Thm. 6.3] for the Casson–Walker invariant of rational homology spheres; see Remark 5.3 for details.

Remark 1.1. — Theorem 1 recovers the third global surgery formula of Lescop [12, Prop. 1.7.8] when restricted to integral surgery coefficients (see Remark 5.2). The proof relies centrally on the "key result" connecting the leading term of the LMO invariant and the Casson–Walker–Lescop invariant [9, 6]. Since both proofs of this key result make use of surgery formulas from [12], it seems necessary to clarify here how the present result is independent from [12, Prop. 1.7.8].

The original proof of [9] indeed makes use of Lescop's third surgery formula. But the alternative proof of [6] uses Lescop's *first* surgery formula, [12, 1.4.8], in terms of the so-called ζ-coefficients extracted from the multivariable Alexander polynomial. In fact, [12, Prop. 1.7.8] is proved in [12, § 6.4] using this first formula.

More importantly, both proofs of the key result use only a very special case of Lescop's formulas: in both [9] and [6], the so-called "diagonalization lemma" (see, e.g. [9, Lem. 6]) is used to reduce the proof to the case of *algebraically split* surgery presentations, that is, for links with zero linking numbers. This means that the key result uses a significantly weaker and simplified version of Lescop's formulas.[1]

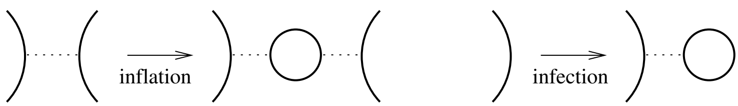
We stress that the core of our first main result is the computation of the leading term of the LMO invariant by purely diagrammatic methods. As a matter of fact, we expect that the techniques developed in this paper could be used in the future to address the next degree term of LMO, which is yet to be understood in general.

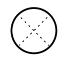
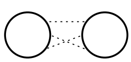
In order to prove Theorem 1, we provide in this paper a number of formulas identifying certain combinations of coefficients of the Kontsevich integral in terms of classical link invariants; see Sections 3 and 4. Such formulas are interesting in themselves, and rather few similar results are known to date,

1. For algebraically split links, our invariant μ_k is just given by $-c_{k+1}$; this is to be compared with the much more involved formulas in Theorem 4.10 and Definition 3.34 for the general case.

see, e.g. [7, 18, 16, 17, 14]. Our formulas are derived from general factorization results which show how certain local configurations in sums of coefficients in the Kontsevich integral yield a factorization by simple link invariants; see Section 3.2.

As an example, the second main result of this paper uses these techniques to give an explicit identification for the z^{n+1}-coefficient c_{n+1} of the Conway polynomial of an n-component link. This identification relies on the definition outlined below of a family of chord diagrams which are recursively built from a couple of low-degree diagrams by simple local operations. Specifically, consider the following two local operations on chord diagrams, called *inflation* and *infection*:

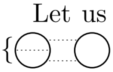
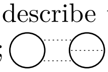
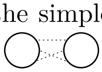
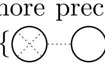
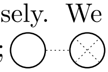


For any integer $n \geq 1$, denote by $\mathcal{E}^-(n)$ the set of all (connected) chord diagrams on n circles which are obtained from the two chord diagrams  and  by iterated inflations in all possible ways. Denote also by $\mathcal{P}(n)$ the set of all diagrams obtained from an element of $\mathcal{E}^-(n-1)$ by a single infection. Our second main result reads as follows:

Theorem 2 (Thm. 3.32). — *Let $n \geq 2$. For any framed oriented n-component link L, we have*

$$c_{n+1}(L) = \sum_{D \in \mathcal{E}^-(n) \cup \mathcal{P}(n)} C_L[D],$$

where $C_L[D]$ denotes the coefficient of D in the (framed) Kontsevich integral of L.

Let us describe the simplest case $n = 2$ more precisely. We have $\mathcal{E}^-(2) = \{$; ; $\}$ and $\mathcal{P}(2) = \{$; $\}$.[2] If L is a framed oriented 2-component link, then Theorem 2 says that $c_3(L)$ is given by

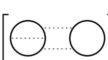
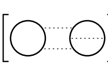
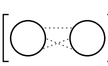
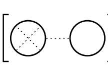
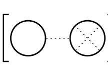
$C_L[$$] + C_L[$$] + C_L[$$] + C_L[$$] + C_L[$$]$.

Figure 1.1 gives typical examples of chord diagrams that are involved in the statement for higher values of n.

2. We use the graphical convention that the circles are ordered from left to right.

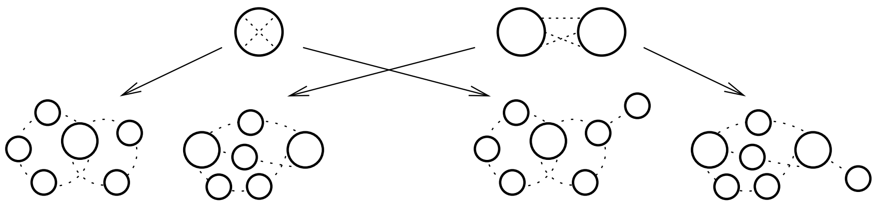


FIGURE 1.1. Two examples of elements in $\mathcal{E}^-(6)$ (left) and $\mathcal{P}(7)$ (right).

The case $n = 1$ is somewhat particular, as it involves a correction term. We have $\mathcal{E}^-(1) = \{$$\}$ and $\mathcal{P}(1) = \emptyset$, and for a knot K we have

$$c_2(K) = C_K\Big[$$  $$\Big] + \frac{1}{24},$$

a formula which is well known to the experts (see Proposition 3.26).

We stress that Theorem 2 is of course related to the weight system of the Conway polynomial, computed in [2] for solving the Melvin–Morton–Rozansky conjecture. Our statement and proof are, however, completely independent of [2].

The paper is organized as follows. In Section 2, we review the various invariants of links and 3-manifolds alluded to in the title of the paper. In Section 3, we identify certain combinations of coefficients in the framed Kontsevich integral in terms of classical invariants; in particular, our factorization results are given in Section 3.2, while Theorem 2 is proved in Section 3.4. Section 4 is devoted to the invariants μ_n; an explicit formula in terms of Conway coefficients and the linking matrix is given in Section 4.2. Finally, we prove Theorem 1 in Section 5.

2. Preliminaries

In this section we recall the necessary material for this paper. We start with a set a conventions that will be used throughout.

2.1. Conventions and notation. — All 3-manifolds will be assumed to be closed, compact, connected and oriented. All links live in the 3-sphere S^3 and are assumed to be framed, oriented and ordered.

Let $L = K_1 \cup \cdots \cup K_n$ be an n-component link. Given a subset I of $\{1, \dots, n\}$, we set

$$L_I := \bigcup_{i \in I} K_i \quad \text{and} \quad L_{\check{I}} := L \setminus L_I.$$

We abbreviate $L_{\check{i}} = L_{\{\check{i}\}}$.