Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 153
Fascicule 2

2025

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 513-558

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 2, tome 153, juin 2025

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 50 € ($ 75)
Abonnement électronique : 165 € ($ 247),
avec supplément papier : Europe 251 €, hors Europe 289 € ($ 433)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directrice de la publication : Isabelle GALLAGHER

Bull. Soc. Math. France
153 (2), 2025, p. 513-558

LOGARITHMIC DIFFERENTIALS ON DISCRETELY RINGED ADIC SPACES

by Katharina Hübner

Abstract. — On a smooth discretely ringed adic space $\mathcal{X}$ over a field k, we define a subsheaf $\Omega^+_{\mathcal{X}}$ of the sheaf of differentials $\Omega_{\mathcal{X}}$. It is defined in a similar way to the subsheaf $\mathcal{O}^+_{\mathcal{X}}$ of $\mathcal{O}_{\mathcal{X}}$ using Kähler seminorms on $\Omega_{\mathcal{X}}$. We give a description of $\Omega^+_{\mathcal{X}}$ in terms of logarithmic differentials. If $\mathcal{X}$ is of the form $\mathrm{Spa}(X, \bar{X})$ for a scheme $\bar{X}$ and an open subscheme X such that the corresponding log structure on $\bar{X}$ is smooth, we show that $\Omega^+_{\mathcal{X}}(\mathcal{X})$ is isomorphic to the logarithmic differentials of $(X, \bar{X})$.

Résumé (*Formes différentielles logarithmique sur espaces adiques discrètement annelés*). — Pour un espace adique lisse et discrètement annelé $\mathcal{X}$ sur un corp k on défine un sous-faisceau $\Omega^+_{\mathcal{X}}$ du faisceau des formes différentielles $\Omega_{\mathcal{X}}$. Il est défini d'une manière similaire au sous-faisceau $\mathcal{O}^+_{\mathcal{X}}$ de $\mathcal{O}_{\mathcal{X}}$ en utilisant la semi-norme de Kähler sur $\Omega_{\mathcal{X}}$. On donne une description de $\Omega^+_{\mathcal{X}}$ en fonction de formes différentielles logarithmiques. Si $\mathcal{X}$ est de la forme $\mathrm{Spa}(X, \bar{X})$ pour un schéma $\bar{X}$ et un sous-schéma ouvert X tel que la structure logarithmique correspondante sur $\bar{X}$ est lisse, on démontre que $\Omega^+_{\mathcal{X}}(\mathcal{X})$ est isomorphe aux formes différentielles logarithmiques de $(X, \bar{X})$.

Texte reçu le 4 mars 2021, modifié le 26 juillet 2024, accepté le 18 septembre 2024.

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Mathematical subject classification (2010). — 14G17, 14A21, 13N05.

Key words and phrases. — Kähler differentials, log geometry, adic spaces, valuation rings.

This research is partly supported by ERC Consolidator Grant 770922 – BirNonArchGeom. Moreover the author acknowledges support by Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 "Geometry and Arithmetic of Uniformized Structures," project number 444845124.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2025/513/$ 5.00
doi:10.24033/bsmf.2905

1. Introduction

Consider a discretely ringed adic space $\mathcal{X}$ over a valued field (k, k^+). Here, discretely ringed means that $\mathcal{X}$ is locally isomorphic to the spectrum of a Huber pair (A, A^+), where A and A^+ carry the discrete topology. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ contains a natural subsheaf $\mathcal{O}^+_{\mathcal{X}}$, the subsheaf of sections with germs of absolute valuation less than or equal to one. One might ask for a similar partner Ω^+ for the sheaf of differentials $\Omega_{\mathcal{X}} = \Omega^1_{\mathcal{X}/k}$. It should be a subsheaf of $\Omega := \Omega^1_{\mathcal{X}}$ defined by a condition $|\omega_x| \leq 1$ for suitable $\mathcal{O}_{\mathcal{X},x}$-seminorms $|\cdot|$ on the stalks $\Omega_{\mathcal{X},x}$ for every point $x \in \mathcal{X}$. Such a sheaf Ω^+ will be useful for investigating cohomological purity for p-torsion sheaves in characteristic $p > 0$. As explained in [16], § 2, the logarithmic de Rham sheaves $\nu(r)$ play a crucial role in cohomological purity. They are defined by an exact sequence

$$0 \to \nu(r) \to \Omega^r_{d=0} \xrightarrow{C-1} \Omega^r \to 0,$$

in the étale topology. Here, "$d = 0$" refers to closed forms and C denotes the Cartier operator (see [15], § 1). However, we expect purity to hold only for the tame topology (see [7] for the definition), and the above sequence is not exact in the tame topology. We hope to solve this problem by replacing Ω^r with $\Omega^{r,+}$. This will be subject to future investigations.

In this article we construct a sheaf Ω^+ as above using the Kähler seminorms (compare [23], § 4.1, for the real valued case) on the stalks Ω_x defined by

$$|\omega|_\Omega := \inf_{\omega = \sum_i f_i dg_i} \max_i \{|f_i| \cdot |g_i|\},$$

where the infimum is taken over all representations of ω as a finite sum $\sum_i f_i dg_i$ (see Section 7.1). However, we need to take care of where we take the infimum. The value group Γ of the valuation on $\mathcal{O}_x$ is not complete in general. We use the new concept of rangers (studied in joint ongoing work of the author with Michael Temkin) to present a construction that serves as completion (see Section 5). As further preparation for the definition of the Kähler seminorm in Section 7.1, we study seminorms taking values in rangers in Section 6. In Section 7.2 we prove that Ω^+ is indeed a sheaf on $\mathcal{X}$. In fact, it is even a sheaf on the tame site $\mathcal{X}_t$ of $\mathcal{X}$ but not on the étale site.

It turns out that Ω^+ has a description in terms of logarithmic differentials. After a preliminary section on the logarithmic cotangent complex (see Section 2), we study logarithmic differentials in Section 4. Let us specify the connection of logarithmic differentials with Ω^+. For a Huber pair (A, A^+) over k such that A is a localization of A^+, we equip A^+ with the log structure $(A^+ \cap A^\times \to A^+)$ on A^+. The corresponding log structure on $\operatorname{Spec} A^+$ is the compactifying log structure associated with the open embedding $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A^+$. The corresponding logarithmic differentials $\Omega^{\log}_{(A,A^+)}$ define a presheaf $\Omega^{\log}$ but not a sheaf. We prove that the sheafification of $\Omega^{\log}$ is Ω^+ in Section 7.2. An important input is that for a local Huber pair (A, A^+) over k, the logarithmic differentials $\Omega^{\log}_{(A,A^+)}$ imbed into Ω_A. For this we need to put some restrictions on (k, k^+). To sum up, we have the following theorem (see Propositions 7.9 and 7.11):

Theorem 1.1. — *Let $\mathcal{X}$ be a discretely ringed adic space over (k, k^+).*

1. *Ω^+ is a sheaf on the tame site $\mathcal{X}_t$.*
2. *Assume that either the residue characteristic of k^+ is zero, k is algebraically closed, or $k = k^+$ is perfect. Then Ω^+ is the Zariski sheafification of the presheaf of logarithmic differentials.*

The last section is dedicated to a study of logarithmic differentials on adic spaces of the form $\mathrm{Spa}(Y, \bar{Y})$, where $\bar{Y}$ is a scheme over the field k and Y is an open subscheme such that the associated log structure on $\bar{Y}$ is log smooth. We call pairs $(Y, \bar{Y})$ of this type log-smooth pairs over k. The main result (Theorem 8.12) is the following:

Theorem 1.2. — *Let $(Y, \bar{Y})$ be log smooth. Then*

$$\Omega^+(\mathrm{Spa}(Y, \bar{Y})) \cong \Omega^{\log}(Y, \bar{Y}),$$

where $\Omega^{\log}$ on the right-hand side is the sheaf of logarithmic differentials on the log scheme associated with $(Y, \bar{Y})$.

The crucial point is that on the adic space $\mathrm{Spa}(Y, \bar{Y})$, we do not need to sheafify $\Omega^{\log}$ in order to compute the global sections of Ω^+. This makes Ω^+ a lot more accessible and it is possible to use the theory of logarithmic differentials on log schemes to investigate Ω^+. We also want to stress that the above isomorphism is obtained without assuming resolution of singularities. The proof relies on the theory of unramified sheaves (see Section 8.2), a notion adapted from [17], and techniques similar to the ones applied in [4] for studying cdh differentials.

2. The logarithmic cotangent complex

On a discretely ringed adic space $\mathcal{X}$, we want to study a subsheaf $\Omega^+_{\mathcal{X}}$ of the sheaf of differentials $\Omega_{\mathcal{X}}$ which is closely related to logarithmic differentials. For future work it will be important to us that this is also a sheaf for the tame topology (not only on the topological space $\mathcal{X}$). For this reason, we need to study the log cotangent complex of a tame extension of valuation rings. The reader not interested in the resulting technical sections 2 and 3 can skip them and jump to Section 4.

In [19] Olsson describes two approaches for a logarithmic cotangent complex. His own construction using log stacks has the advantage that it is trivial for log smooth morphisms. However, transitivity triangles only exist under certain conditions and the construction only works for fine log schemes, i.e., under strong finiteness conditions that are not satisfied in our situation. Gabber's version described in [19], §8, is more functorial, but it has the disadvantage that it is not trivial for all log smooth morphisms. We will use Gabber's log cotangent complex and compare it in special situations to Olsson's in order to make explicit computations. Slightly more generally, we will define the log cotangent complex for simplicial prelog rings as described, for instance, in [1], §5, or [21], §4.

Let us start by reviewing some definitions. Recall that a prelog ring is a ring R and a (commutative) monoid M together with a homomorphism of monoids $M \to R$, where R is considered as a monoid with its multiplicative structure. A log ring is a prelog ring $\iota : M \to R$ inducing an isomorphism $\iota^{-1}(R^\times) \to R^\times$. The inclusion of the category of log rings into prelog rings has a left adjoint called logification (see [18], Chapter II, Proposition 1.1.5). We write $(M^a \to R)$ or $(M \to R)^a$ for the logification of $(M \to R)$.

Denote by Set, Mon, Ring, and $\mathrm{LogRing}^{\mathrm{pre}}$ the categories of sets, monoids, rings, and prelog rings, respectively. We write sSet, sMon, sRing, and $s\mathrm{LogRing}^{\mathrm{pre}}$ for the respective categories of simplicial objects. We endow sSet with the standard model structure, i.e., the weak equivalences are the maps inducing a weak homotopy equivalence on geometric realizations and the fibrations are the Kan fibrations. Defining the (trivial) fibrations to be the homomorphisms that are (trivial) fibrations on the underlying category of simplicial sets, we obtain a closed model structure on sRing and sMon (see [1], §4). Now consider the forgetful functor

$$\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}} : s\mathrm{LogRing}^{\mathrm{pre}} \longrightarrow s\mathrm{Mon} \times s\mathrm{Ring}$$

mapping $(M \to A)$ to (M, A). By [21], Proposition 3.3, there is a projective proper simplicial cellular model structure on $s\mathrm{LogRing}^{\mathrm{pre}}$, whose fibrations and weak equivalences are the maps that are mapped to fibrations and weak equivalences, respectively, under $\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}}$. With respect to this model structure, $\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}}$ is a left and right Quillen functor ([1], Propositions 5.3 and 5.5). Its left adjoint is the functor $\mathrm{Free}^{s\mathrm{Mon}\times s\mathrm{Ring}}_{s\mathrm{LogRing}^{\mathrm{pre}}}$ mapping (M, A) to $(M \to A[M])$.

For a homomorphism $(M \to A) \to (N \to B)$ of simplicial prelog rings, we write $s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)}$ for the category of simplicial $(M \to A)$-algebras over $(N \to B)$. It inherits a model structure from $s\mathrm{LogRing}^{\mathrm{pre}}$. Consider the functor

$$\begin{aligned}\Omega : s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)} &\to \mathrm{Mod}_B\\ (L \to C) &\mapsto \Omega^1_{(L\to C)/(M\to A)} \otimes_C B,\end{aligned}$$

where Ω^1 is defined by applying to each level the functor of log Kähler differentials (see [18], Chapter IV, Proposition 1.1.2; note that a log ring in loc. cit. is what we here call a prelog ring). Being a left Quillen functor ([21], Lemma 4.6), it has a left derived functor

$$L\Omega : \mathrm{Ho}(s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)}) \to \mathrm{Ho}(\mathrm{Mod}_B)$$

on the respective homotopy categories. The image of $(N \to B)$ under $L\Omega$ is called the *cotangent complex* of $(N \to B)$ and denoted $\mathbb{L}_{(N\to B)/(M\to A)}$. For a homomorphism $(M \to A) \to (N \to B)$ of discrete log rings, it can be computed as follows: for shortness write $F := \mathrm{Forget}^{\mathrm{LogRing}^{\mathrm{pre}}}_{\mathrm{Mon}\times\mathrm{Ring}}$ and $G := \mathrm{Free}^{\mathrm{Mon}\times\mathrm{Ring}}_{\mathrm{LogRing}^{\mathrm{pre}}}$ (the discrete versions of the above-considered functors). We have a canonical free resolution

$$(1)\qquad \ldots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; GFGF(N\to B) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; GF(N\to B) \;\substack{\longleftarrow\\ \longrightarrow}\; (N\to B),$$

which we denote by $P_\bullet \to (N \to B)$. Then $\mathbb{L}_{(N\to B)/(M\to A)}$ is represented by $\Omega(P_\bullet)$. In particular, we recover Gabber's definition ([19], Definition 8.5).

The cotangent complex has the following two important properties (see [21], Proposition 4.12):

Proposition 2.1. — (i) Transitivity. *Let* $(M \to A) \to (N \to B) \to (K \to C)$ *be maps of simplicial prelog rings. Then there is a homotopy cofiber sequence in* $\mathrm{Ho}(\mathrm{Mod}_C)$

$$C \otimes^h_B \mathbb{L}_{(N\to B)/(M\to A)} \to \mathbb{L}_{(K\to C)/(M\to A)} \to \mathbb{L}_{(K\to C)/(N\to B)}.$$

(ii) Base change. *Let*

$$\begin{array}{ccc} (N'\to B') & \longleftarrow & (N\to B) \\ \uparrow & & \uparrow \\ (M'\to A') & \longleftarrow & (M\to A) \end{array}$$

be a homotopy pushout square in $s\mathrm{LogRing}^{\mathrm{pre}}$*. Then there is an isomorphism in* $\mathrm{Ho}(\mathrm{Mod}_{B'})$

$$B' \otimes^h_B \mathbb{L}_{(N\to B)/(M\to A)} \cong \mathbb{L}_{(N'\to B')/(M'\to A')}.$$

In order to apply these results in our setting of discrete prelog rings, it would be useful to know when the homotopy pushouts appearing in (i) and (ii) coincide with the ordinary pushouts. The homotopy pushout in (i) appearing in the cofiber sequence is taken in the homotopy category of Mod_C. Suppose that C and B are discrete. Then it is well known that

$$C \otimes_B \mathbb{L}_{(N\to B)/(M\to A)} \cong C \otimes^h_B \mathbb{L}_{(N\to B)/(M\to A)}$$

in the case that C is flat over B. In the base change setting for discrete prelog rings, it turned out to be easier to prove the base change result from