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AN APPLICATION OF p-ADIC INTEGRATION TO THE DYNAMICS OF A BIRATIONAL TRANSFORMATION PRESERVING A FIBRATION

BY FEDERICO LO BIANCO

ABSTRACT. — Let $f\colon X \dashrightarrow X$ be a birational transformation of a projective manifold X whose Kodaira dimension $\kappa(X)$ is non-negative. We show that if there exist a dominant rational map $\pi\colon X \dashrightarrow B$ and a birational transformation $f_B\colon B \dashrightarrow B$ which preserves a big line bundle $L \in \mathrm{Pic}(B)$ such that $f_B \circ \pi = \pi \circ f$, then f_B has finite order.

As a corollary, we show that for projective irreducible symplectic manifolds of type $K3^{[n]}$ or generalized Kummer, the first dynamical degree characterizes the birational transformations admitting a Zariski-dense orbit.

RÉSUMÉ (*Une application de l'intégration p-adique à la dynamique d'une transformation birationnelle préservant une fibration*). — Soit $f\colon X \dashrightarrow X$ une transformation birationnelle d'une variété projective lisse X dont la dimension de Kodaira $\kappa(X)$ est non-négative. Nous montrons que, s'il existe une application rationnelle dominante $\pi\colon X \dashrightarrow B$ et une transformation birationnelle $f_B\colon B \dashrightarrow B$ qui préserve un fibré en droites "big" $L \in \mathrm{Pic}(B)$ et telle que $f_B \circ \pi = \pi \circ f$, alors f_B est d'ordre fini.

Comme corollaire, nous montrons que, pour toute variété projective symplectique irréductible de type $K3^{[n]}$ ou Kummer généralisée, le premier degré dynamique caractérise les transformations birationnelles admettant une orbite Zariski-dense.

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1. Introduction

Let $f\colon X \dashrightarrow X$ be a birational transformation of a complex projective manifold. A natural question when studying the dynamical properties of f is the existence of an equivariant rational fibration, i.e. of a dominant rational map with connected fibres $\pi\colon X \dashrightarrow B$ onto a projective manifold and of a birational transformation $f_B\colon B \dashrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \overset{f}{\dashrightarrow} & M \\ {\scriptstyle\pi}\,\vdots & & \vdots\,{\scriptstyle\pi} \\ B & \overset{f_B}{\dashrightarrow} & B \end{array} .$$

The transformation f is called *imprimitive* (see [5]) if there exists a non-trivial f-equivariant fibration (i.e. such that $0 < \dim B < \dim X$) and is primitive otherwise; the study of imprimitive birational transformations should intuitively be simpler than of primitive ones, as their dynamics decomposes into smaller dimensional dynamical systems: the base and the fibres. The goal of the present paper is to study the action on the base induced by an imprimitive transformation. When π is (birationally equivalent to) the canonical fibration of X, some finite index subgroup of $\mathrm{Bir}(X)$ acts as the identity on B; this is a consequence of the finiteness of the pluricanincal representations (see [17]). Our main result is the following generalization:

THEOREM A. — *Let X be a complex projective manifold and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a dominant rational map $\pi\colon X \dashrightarrow B$ onto a projective manifold B and a birational transformation $f_B\colon B \dashrightarrow B$ such that $f_B \circ \pi = \pi \circ f$. Assume that*

1. *the Kodaira dimension $\kappa(X)$ of X is non-negative;*
2. *f_B preserves a big line bundle L.*

Then f_B has finite order.

Following [2], we say that a birational transformation $g\colon Y \dashrightarrow Y$ preserves a line bundle L on Y if there exists a resolution of indeterminacies

$$\begin{array}{ccc} \widetilde{Y} & & \\ {\scriptstyle\eta}\downarrow & \searrow^{\widetilde{g}} & \\ Y & \overset{g}{\dashrightarrow} & Y \end{array}$$

such that $\eta^* L = \widetilde{g}^* L$; when g is an automorphism, or more generally a pseudo-automorphism (i.e. g and g^{-1} do not contract any hypersurface), one can define the pull-back of a line bundle through g and show that L is preserved by g if and only if $g^* L = L$. A birational transformation g which preserves a line bundle L induces a linear automorphism

$$g^*\colon H^0(Y,L)\to H^0(Y,L)$$

defined by $g^*(s):=\eta_*\widetilde{g}^*(s)$; this definition makes sense because by the projection formula, $\eta_*\eta^*L=L$.

Remark 1.1. — The second assumption of Theorem A is automatically verified if $\pm K_B$ is big and g is a pseudo-automorphism; note, however, that if K_B is big, i.e. B is of general type, then the group of birational transformations is finite, which implies the conclusion of the theorem.

If X is irreducible symplectic and $\pi\colon X\to B$ is a regular fibration onto a smooth projective manifold, then K_B is ample, i.e. B is Fano (see [12, Corollary 1.3]).

By the same approach, we also obtain an analogous result concerning groups of transformations. Recall that the group $\mathrm{Aut}(X)$ of automorphisms of X can be naturally seen as a Zariski-open subset of the Hilbert scheme of subvarieties of $X\times X$, which endows it with a natural topology; we denote by $\mathrm{Aut}_0(X)$ the connected component of the identity of $\mathrm{Aut}(X)$ and by $\mathrm{Bir}(X)$ the group of birational transformations of X.

Theorem B. — *Let X be a complex projective manifold with $\kappa(X)\geq 0$ and let $G\subset \mathrm{Bir}(X)$ be a group of birational transformations of X. Suppose that there exist a dominant rational map $\pi\colon X \dashrightarrow B$ onto a projective manifold B which is preserved by G and let π_*G denote the image of the natural morphism $\pi_*\colon G\to \mathrm{Bir}(B)$. Assume that*

1. *the quotient $G/(G\cap \mathrm{Aut}_0(X))$ is finitely generated;*
2. *all elements of π_*G preserve a big line bundle L.*

Then π_*G is finite.

Proof. — The Kodaira–Iitaka fibration associated with some multiple $L^{\otimes h}$ of L is birational onto its image and allows us to identify π_*G with a subgroup of $\mathrm{PGL}\, H^0(B,L^{\otimes h})=\mathrm{PGL}_{N+1}(\mathbb{C})$; see §4.2 for details. By Theorem A, π_*G is torsion.

The connected component $\mathrm{Aut}_0(X)$ has a natural structure of a connected algebraic group; therefore, by Chevalley's structure theorem (see for example [13, Theorem 8.27]), it is isomorphic to an extension of an abelian variety by a linear algebraic group. Since $\kappa(X)\geq 0$, linear algebraic groups have trivial action on X by [17, Theorem 14.1]; this means that $\mathrm{Aut}_0(X)$ is an abelian variety.

Let $G_0:=G\cap\mathrm{Aut}_0(X)$; note that the topological closure $\overline{G}_0\leq \mathrm{Aut}_0(X)$ still preserves the fibration π and that the induced action on B preserves L (this is a consequence of the seesaw theorem, see [14, Corollary 5.6]). In particular, π_* extends to $\overline{G}_0$ and, since $\overline{G}_0$ is compact, so is its image $\pi_*\overline{G}_0$.

Again by Theorem A, $\pi_*\overline{G}_0$ is torsion. Compact torsion subgroups of Lie groups are finite by [10], thus $\pi_*\overline{G}_0$ is finite; hence, a fortiori, so is $\pi_* G_0$.

Let $K \lhd G_0$ denote the kernel of the restriction $\pi_*\colon G_0 \to \mathrm{PGL}_{N+1}(\mathbb{C})$; since K has finite index in G_0, the group G/K is finitely generated. The homomorphism π_* factors through a natural homomorphism

$$\phi\colon G/K \to \mathrm{PGL}_{N+1}(\mathbb{C}),$$

so that $\pi_* G = \phi(G/K)$.

By [10], a torsion subgroup of a Lie group is virtually abelian; furthermore, by Schreier's lemma, a finite index subgroup of a finitely generated group is finitely generated. Since it is easy to show that an abelian, finitely generated torsion subgroup of $\mathrm{PGL}_{N+1}(\mathbb{C})$ is finite, we obtain that a finite index subgroup of $\phi(G/K)$ is finite. In particular, $\pi_* G = \phi(G/K)$ is finite, which concludes the proof. □

The following corollary has the advantage of requiring only a numerical hypothesis on the action of f_B, instead of having to compute its action on the Picard group.

Corollary C. — *Let X be a projective manifold and let $f\colon X \dashrightarrow X$ be a birational transformation. Suppose that there exist a dominant rational map $\pi\colon X \dashrightarrow B$ onto a projective manifold B and a birational transformation $f_B\colon B \dashrightarrow B$ such that $f_B \circ \pi = \pi \circ f$. Assume that*

1. *the Kodaira dimension $\kappa(X)$ is non-negative;*
2. $\mathrm{Pic}^0(B) = 0$*;*
3. *the induced linear maps $(f_B^N)^*\colon H^*(B,\mathbb{C}) \to H^*(B,\mathbb{C})$ have bounded norm as $N \to +\infty$.*

Then f_B has finite order.

Remark 1.2. — The second and third assumptions of Corollary C are automatically satisfied if $\mathrm{Pic}^0(X) = 0$ and the induced linear maps $(f^N)^*\colon H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ have bounded norm.

Proof. — Since the induced linear maps $(f_B^N)^*\colon H^*(B,\mathbb{C}) \to H^*(B,\mathbb{C})$ have a bounded norm as $N \to +\infty$, by Weil's regularization theorem (see [6, Theorem 3]), up to replacing B by a smooth birational model and f_B by an iterate, we may assume that f_B is an automorphism and that $f_B \in \mathrm{Aut}^0(B)$. In particular, f_B has trivial action on $H^*(B,\mathbb{C})$ and, thus, since line bundles on B are uniquely determined by their numerical class, on $\mathrm{Pic}(B)$. Therefore, f_B is an automorphism which preserves an ample line bundle, hence by Theorem A, it has finite order. □

An almost equivalent formulation in terms of Kodaira–Iitaka fibrations is the following:

Corollary D. — *Let X be a complex projective manifold with non-negative Kodaira dimension and let $f\colon X \dashrightarrow X$ be a birational transformation. If f preserves a line bundle $\mathcal{L}$, then the induced projective automorphism*

$$f^*\colon \mathbb{P}H^0(X,\mathcal{L}) \to \mathbb{P}H^0(X,\mathcal{L})$$

has finite order.

As was pointed out to me by Vlad Lazić, such a formulation is linked with the problem of determining the finiteness of pluri-log-canonical representations, which in turn plays a role in the problem of abundance conjecture. In this context, one needs to consider the linear action of the group of birational transformations of a normal scheme X preserving a divisor Δ on the space of sections $H^0(X, m(K_X+\Delta))$ (instead of its projectification, as is done in the present work). Proving the finiteness of the linear action for all m multiples of a certain m_0 (which is a stronger result than Corollary D), together with the application of the minimal model programme, allows the abundance conjecture to be reduced to the case of log-canonical pairs (see [2, Theorem 5.10] and [3, Theorem 1.4]). See [2, 3] for more details and for positive results in this direction.

Remark 1.3. — Using Theorem B, it is not hard to extend Corollary C and Corollary D to groups G of birational transformations such that $G/(G\cap \mathrm{Aut}_0(X))$ is finitely generated.

1.1. The case of irreducible symplectic manifolds. — Theorem A is particularly interesting in the case where X is an irreducible symplectic manifold. A compact Kähler manifold is said to be *irreducible symplectic* (or hyperkähler) if it is simply connected and the vector space of holomorphic 2-forms is spanned by a nowhere degenerate form. Irreducible symplectic manifolds form, together with Calabi–Yau manifolds and complex tori, one of the three fundamental classes of compact Kähler manifolds with trivial Chern class.

Example 1.4. — All $K3$ surfaces are irreducible symplectic; more generally, if S is a $K3$ surface, then the Hilbert scheme $S^{[n]}$ of n points on S is an irreducible symplectic manifold of dimension $2n$.

Similarly, if T is a two-dimensional complex torus and if

$$\Sigma\colon T^{[n+1]} \to T$$

denotes the sum morphism, i.e. $\Sigma[p_1,\dots,p_{n+1}] := \sum_i p_i \in T$, then any fibre of Σ is an irreducible symplectic manifold of dimension $2n$, called the generalized Kummer variety associated to T.