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INTERMEDIATE JACOBIANS AND RATIONALITY OVER ARBITRARY FIELDS

BY OLIVIER BENOIST AND OLIVIER WITTENBERG

ABSTRACT. — We prove that a three-dimensional smooth complete intersection of two quadrics over a field k is k -rational if and only if it contains a line defined over k . To do so, we develop a theory of intermediate Jacobians for geometrically rational threefolds over arbitrary, not necessarily perfect, fields. As a consequence, we obtain the first examples of smooth projective varieties over a field k which have a k -point, and are rational over a purely inseparable field extension of k , but not over k .

RÉSUMÉ. — Nous démontrons qu'une intersection complète lisse de deux quadriques de dimension 3 sur un corps k est k -rationnelle si et seulement si elle contient une droite définie sur k . À cet effet, nous développons une théorie des jacobiniennes intermédiaires pour les variétés géométriquement rationnelles de dimension 3 sur des corps quelconques, non nécessairement parfaits. Comme conséquence, nous obtenons les premiers exemples de variétés projectives lisses sur un corps k qui ont un k -point, et sont rationnelles sur une extension de corps purement inséparable de k , mais pas sur k .

Introduction

Let k be a field. A variety X of dimension n over k is said to be k -rational (resp. k -unirational, resp. separably k -unirational) if there exists a birational map (resp. a dominant rational map, resp. a dominant and separable rational map) $\mathbf{A}_k^n \dashrightarrow X$.

This article is devoted to studying the k -rationality of threefolds over k . Our main result answers positively a conjecture of Kuznetsov and Prokhorov.

THEOREM A (Theorem 4.7). — *Let $X \subset \mathbf{P}_k^5$ be a smooth complete intersection of two quadrics. Then X is k -rational if and only if it contains a line defined over k .*

The question of the validity of Theorem A goes back to Auel, Bernardara and Bolognesi [5, Question 5.3.2 (3)], who raised it when k is a rational function field in one variable over an algebraically closed field.

Using the fact that varieties X as in Theorem A are separably k -unirational if and only if they have a k -point (see Theorem 4.8), we obtain new counterexamples to the Lüroth problem over non-closed fields.

THEOREM B (Theorem 4.14). – *For any algebraically closed field κ , there exists a three-dimensional smooth complete intersection of two quadrics $X \subset \mathbf{P}_{\kappa((t))}^5$ which is separably $\kappa((t))$ -unirational, $\kappa((t^{\frac{1}{2}}))$ -rational, but not $\kappa((t))$ -rational.*

When κ has characteristic 2, Theorem B yields the first examples of smooth projective varieties over a field k which have a k -point and are rational over the perfect closure of k , but which are not k -rational (see Remarks 4.15 (iii) and (iv)).

Theorem A may be compared to the classical fact that a smooth quadric over k is k -rational if and only if it has a k -point. However, although it is easy to check that a smooth projective k -rational variety has a k -point, the fact that a k -rational three-dimensional smooth complete intersection of two quadrics X necessarily contains a k -line is highly non-trivial. To prove it, we rely on obstructions to the k -rationality of X arising from a study of its intermediate Jacobian.

Such obstructions go back to the seminal work of Clemens and Griffiths [15]: if a smooth projective threefold over $\mathbf{C}$ is $\mathbf{C}$ -rational, then its intermediate Jacobian is isomorphic, as a principally polarized abelian variety over $\mathbf{C}$, to the Jacobian of a (not necessarily connected) smooth projective curve. This implication was used in [15] to show that smooth cubic threefolds over $\mathbf{C}$ are never $\mathbf{C}$ -rational, and was later applied to show the irrationality of several other classes of complex threefolds (see for instance [6]). The work of Clemens and Griffiths was extended by Murre [71] to algebraically closed fields of any characteristic different from 2.

More recently, we implemented the arguments of Clemens and Griffiths over arbitrary perfect fields k [7]. By exploiting the fact that the intermediate Jacobian may be isomorphic to the Jacobian of a smooth projective curve over $\bar{k}$ while not being so over k , we produced new examples of varieties over k that are $\bar{k}$ -rational but not k -rational.

Hassett and Tschinkel [46] subsequently noticed that over a non-closed field k , the intermediate Jacobian carries further obstructions to k -rationality: if X is a smooth projective k -rational threefold, then not only is its intermediate Jacobian isomorphic to the Jacobian $\mathbf{Pic}^0(C)$ of a smooth projective curve C over k , but in addition, assuming for simplicity that C is geometrically connected of genus ≥ 2 , the $\mathbf{Pic}^0(C)$ -torsors associated with $\mathrm{Aut}(\bar{k}/k)$ -invariant algebraic equivalence classes of codimension 2 cycles on X are of the form $\mathbf{Pic}^\alpha(C)$ for some $\alpha \in \mathbf{Z}$. When X is a smooth three-dimensional complete intersection of two quadrics, they used these obstructions in combination with the natural identification of the variety of lines of X with a torsor under the intermediate Jacobian of X , and with the work of Wang [82], to prove Theorem A when $k = \mathbf{R}$ [46, Theorem 36] (and later [45] over subfields of $\mathbf{C}$).

The aim of the present article is to extend these arguments to arbitrary fields.

Applications to k -rationality criteria for other classes of $\bar{k}$ -rational threefolds appear in the work of Kuznetsov and Prokhorov [61].

So far, we have been imprecise about what we call the intermediate Jacobian of a smooth projective threefold X over k .

If $k = \mathbf{C}$, one can use Griffiths' intermediate Jacobian $J^3 X$ constructed by transcendental means. This is the original path taken by Clemens and Griffiths [15]. The algebraic part of Griffiths' intermediate Jacobian has been shown to descend to subfields $k \subseteq \mathbf{C}$ by Achter, Casalaina-Martin and Vial [3, Theorem B]; the resulting k -structure on $J^3 X$ is the one used in [45].

Over algebraically closed fields k of arbitrary characteristic, a different construction of an intermediate Jacobian $\mathrm{Ab}^2(X)$, based on codimension 2 cycles, was provided by Murre [72, Theorem A p. 226] (see also [54]). This cycle-theoretic approach to intermediate Jacobians had already been applied by him to rationality problems (see [71]).

Over a perfect field k , the universal property satisfied by Murre's intermediate Jacobian $\mathrm{Ab}^2(X_{\bar{k}})$ induces a Galois descent datum on $\mathrm{Ab}^2(X_{\bar{k}})$, thus yielding a k -form $\mathrm{Ab}^2(X)$ of $\mathrm{Ab}^2(X_{\bar{k}})$ [3, Theorem 4.4]. It is this intermediate Jacobian $\mathrm{Ab}^2(X)$, which coincides with $J^3 X$ when $k \subseteq \mathbf{C}$, that we used in [7].

Over an imperfect field k , one runs into the difficulty that Murre's definition of $\mathrm{Ab}^2(X_{\bar{k}})$ does not give rise to a $\bar{k}/k$ -descent datum on $\mathrm{Ab}^2(X_{\bar{k}})$. Achter, Casalaina-Martin and Vial still prove, in [4], the existence of an algebraic representative $\mathrm{Ab}^2(X)$ for algebraically trivial codimension 2 cycles on X (see §1.2 of *op. cit.* for the definition). However, when k is imperfect, it is not known whether $\mathrm{Ab}^2(X)_{\bar{k}}$ is isomorphic to $\mathrm{Ab}^2(X_{\bar{k}})$. For this reason, we do not know how to construct on $\mathrm{Ab}^2(X)$ the principal polarization that is so crucial to the Clemens-Griffiths method.

To overcome this difficulty and prove Theorem A in full generality, we provide, over an arbitrary field k , an entirely new construction of an intermediate Jacobian.

Our point of view is inspired by Grothendieck's definition of the Picard scheme (for which see [33], [12, Chapter 8], [56]). With any smooth projective $\bar{k}$ -rational threefold X over k , we associate a functor $\mathrm{CH}_{X/k, \mathrm{fppf}}^2 : (\mathrm{Sch}/k)^{\mathrm{op}} \rightarrow (\mathrm{Ab})$ endowed with a natural bijection $\mathrm{CH}^2(X_{\bar{k}}) \xrightarrow{\sim} \mathrm{CH}_{X/k, \mathrm{fppf}}^2(\bar{k})$ (see Definition 2.9 and (3.1)). The functor $\mathrm{CH}_{X/k, \mathrm{fppf}}^2$ is an analogue, for codimension 2 cycles, of the Picard functor $\mathrm{Pic}_{X/k, \mathrm{fppf}}$.

Too naive attempts to define the functor $\mathrm{CH}_{X/k, \mathrm{fppf}}^2$ on the category of k -schemes, such as the formula " $T \mapsto \mathrm{CH}^2(X_T)$ ", fail as Chow groups of possibly singular schemes are not even contravariant with respect to arbitrary morphisms: one would need to use a contravariant variant of Chow groups (see Remark 3.2 (ii)). To solve this issue, we view Chow groups of codimension ≤ 2 as subquotients of K -theory by means of the Chern character, and we define $\mathrm{CH}_{X/k, \mathrm{fppf}}^2$ as an appropriate subquotient of (the fppf sheafification of) the functor $T \mapsto K_0(X_T)$. That this procedure gives rise to the correct functor, even integrally, is a consequence of the Riemann-Roch theorem without denominators [53].

We show that $\mathrm{CH}_{X/k, \mathrm{fppf}}^2$ is represented by a smooth k -group scheme $\mathbf{CH}_{X/k}^2$ (Theorem 3.1 (i)). Our functorial approach is crucial for this, as it allows us to argue by fppf descent from a possibly inseparable finite extension l of k such that X is l -rational. By construction, there is a natural isomorphism $\mathbf{CH}_{X_l/l}^2 \simeq (\mathbf{CH}_{X/k}^2)_l$ for all field extensions l of k .