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THE MODULI OF ANNULI IN RANDOM CONFORMAL GEOMETRY

BY Morris ANG, Guillaume REMY AND Xin SUN

ABSTRACT. – We obtain exact formulae for three basic quantities in random conformal geometry that depend on the modulus of an annulus. The first is for the law of the modulus of the Brownian annulus describing the scaling limit of uniformly sampled planar maps with annular topology, which is as predicted from the ghost partition function in bosonic string theory. The second is for the law of the modulus of the annulus bounded by a loop of a simple conformal loop ensemble (CLE) on a disk and the disk boundary. The formula is as conjectured from the partition function of the O(n) loop model on the annulus derived by Saleur-Bauer (1989) and Cardy (2006). The third is for the annulus partition function of the $\mathrm{SLE}_{8/3}$ loop introduced by Werner (2008), confirming another prediction of Cardy (2006). The physics principle underlying our proofs is that 2D quantum gravity coupled with conformal matters can be decomposed into three conformal field theories (CFT): the matter CFT, the Liouville CFT, and the ghost CFT. At the technical level, we rely on two types of integrability in Liouville quantum gravity, one from the scaling limit of random planar maps, the other from the Liouville CFT.

RÉSUMÉ. – Dans cet article nous obtenons des formules exactes pour trois quantités de base en géométrie conforme aléatoire qui dépendent du module d'un anneau. La première concerne la loi du module de l'anneau brownien décrivant la limite d'échelle des cartes planaires aléatoires uniformes avec la topologie de l'anneau, comme prédit par la fonction de partition de la théorie des cordes bosoniques. La seconde concerne la loi du module de l'anneau délimité par une boucle du « conformal loop ensemble » (CLE) dans le disque et par le bord du disque. La formule a été conjecturée à partir de la fonction de partition du modèle de boucle O(n) sur l'anneau obtenue par Saleur-Bauer (1989) et Cardy (2006). La troisième formule concerne la fonction de partition sur l'anneau de l'ensemble de boucle $\mathrm{SLE}_{8/3}$ introduite par Werner (2008), et confirmant une autre prédiction de Cardy (2006). Le principe physique qui sous-tend nos preuves est que la gravité quantique 2D couplée à un champ de matière conforme peut être décomposée en trois théories conformes des champs (CFT) : la CFT des champs de matière, la CFT de Liouville et la CFT fantôme. Les techniques de preuve utilisent deux types d'intégrabilité dans la gravité quantique de Liouville, l'une à partir de la limite d'échelle des cartes planaires aléatoires, et l'autre venant de la CFT de Liouville.

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1. Introduction

Over the last two decades, tremendous progress has been made in understanding random surfaces of simply connected topology. By [49, 54, 16] and their extensions, the uniformly sampled random planar maps on the sphere and the disk converge as metric-measure spaces in the scaling limit. The limiting surfaces are known as the Brownian sphere and disk, respectively. Moreover, as proved in various senses [57, 58, 59, 41, 42], once these surfaces are conformally embedded in the complex plane, the embedded random geometry is given by Liouville quantum gravity (LQG) [31, 29, 40] with $\gamma = \sqrt{8/3}$. Finally, as shown in [7, 22, 5], the law of the random field inducing the random geometry is given by the Liouville conformal field theory on the sphere [25] and the disk [43]. For random planar maps decorated with statistical physics models, the scaling limit is described by LQG surfaces decorated with Schramm Loewner evolution (SLE) and conformal loop ensemble (CLE). Although there are still major open questions concerning the scaling limit, the picture in the continuum is well understood, thanks to the theory of the quantum zipper [72] and the mating-of-trees [30]. For more background on this subject, see the surveys [37, 73].

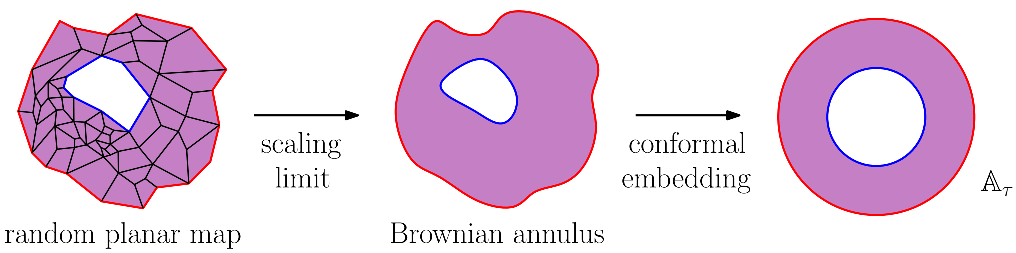

FIGURE 1. Conformal embedding of the Brownian annulus to the annulus $\mathbb{A}_\tau = \{z : |z| \in (e^{-2\pi\tau}, 1)\}$ with (random) modulus τ

For non-simply-connected surfaces, the key new feature is the non-uniqueness of the conformal structure. For example, if we conformally embed the Brownian annulus to the complex plane (Figure 1), the modulus of the planar annulus is random. Precise descriptions of these surfaces, including the laws of the random moduli, were conjectured in [26] for the torus, in [35] for higher genus surfaces, and in [63] by the second named author for the annulus. The conjecture is based on a fundamental principle that dates back to Polyakov's seminal work [62] on bosonic string theory. Namely, 2D quantum gravity coupled with conformal matter can be decomposed into three conformal field theories (CFT): the matter CFT, the Liouville CFT, and the ghost CFT. In this paper, we prove this conjecture for annulus in the case of pure gravity which corresponds to the Brownian annulus, and the cases where the matter corresponds to the simple CLE [71, 75] (i.e., CLE_κ with $\kappa \in (8/3, 4]$) or Werner's $\mathrm{SLE}_{8/3}$ loop [78].

We also derive some exact formulae for SLE/CLE conjectured from physics. Since the discovery of SLE, it has been proved or widely conjectured that SLE/CLE describe the scaling limits of interfaces in 2D lattice models such as percolation and the Ising model. Many scaling exponents/dimensions of such models predicted by CFT [10] were derived rigorously from SLE/CLE (e.g., [48]). Moreover, some exact formulae predicted via boundary CFT, such as Cardy's formula for percolation crossing, were rigorously derived from SLE (see e.g., [77]). Saleur-Bauer [67] and Cardy [21] derived a formula for the partition function of the $O(n)$ loop model on the annulus, which provides crucial information on the CFT description of its scaling limit. Based on the conjectural relation between CLE and the $O(n)$ loop model (see e.g., [71]), this partition function corresponds to an exact formula for the loop statistics of CLE. Viewing self avoiding loop as the $O(0)$ model, Cardy [21] further conjectured a formula for the annulus partition function of the $\mathrm{SLE}_{8/3}$ loop. Deriving these formulae for SLE/CLE has been an open question. In this paper we prove them for simple CLE and the $\mathrm{SLE}_{8/3}$ loop.

The key to our proofs is an explicit relation between the partition function of LQG surfaces and the law of their moduli. This is obtained from the conformal bootstrap of Liouville CFT on the annulus due to Wu [79]. It can be viewed as a Knizhnik-Polyakov-Zamolodchikov (KPZ) relation [45] between the quantum and the Euclidean geometry at the partition function level, whereas the traditional KPZ relation established in probability [31] is at the level of scaling exponents/dimensions. Our KPZ relation reduces the computation of an annular quantity depending on the modulus to the same quantity for LQG surfaces, which is easier to obtain from the random planar map perspective.

Our work represents the first step towards understanding the relation between LQG in the random geometry framework and measures on the moduli space of Riemann surfaces, although quantum gravity intuition has inspired important developments of the latter subject since 1980s. Our work is also a further step towards uncovering the CFT content of CLE, after the proof of the imaginary DOZZ formula by the first and third named authors together with Gefei Cai and Baojun Wu [1]. It continues to demonstrate the rich interplay between various kinds of integrability in conformal probability, a theme recently explored in [5, 6, 1, 2]. We expect that methods in our paper together with the remarkable developments in the integrability of Liouville CFT [46, 34] will result in further progress in this direction.

In the rest of the introduction, we first state our result on the modulus of the Brownian annulus in Section 1.1. Then we present the matter-Liouville-ghost decomposition for the Brownian annulus and the CLE decorated quantum annulus in Section 1.2. We state our KPZ relation for annulus partition functions in Section 1.3 and results for simple CLE and the $\mathrm{SLE}_{8/3}$ loop in Sections 1.4 and 1.5, respectively. In Section 1.6 we discuss some subsequent projects and open questions concerning other models or topologies.

Notation for the modulus. – Let $\mathbb{C}$ be the complex plane. For $\tau \in (0, \infty)$, let $\mathbb{A}_\tau = \{z \in \mathbb{C} : |z| \in (e^{-2\pi\tau}, 1)\}$. For a Riemann surface A of the annular topology, there exists a unique τ such that A and $\mathbb{A}_\tau$ are conformally equivalent. We call τ the *modulus* of A.

1.1. Random moduli for the Brownian annulus

There are various equivalent ways of introducing the Brownian annulus. For the convenience of describing its conformal structure, we will introduce it via the Brownian disk. For $a > 0$, the *Brownian disk with boundary length a* is a random metric-measure space with the disk topology, which is the Gromov-Hausdorff-Prokhorov (GHP) scaling limit of random quadrangulations or triangulations of a large polygon under the critical Boltzmann