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# FOLIATED PLATEAU PROBLEMS AND ASYMPTOTIC COUNTING OF SURFACE SUBGROUPS

BY SÉBASTIEN ALVAREZ, BEN LOWE AND GRAHAM SMITH

ABSTRACT. – In 2000, Labourie initiated the study of the dynamical properties of the space of $k$-surfaces, that is, suitably complete immersed surfaces of constant extrinsic curvature in 3-dimensional manifolds, which he presented as a higher-dimensional analogue of the geodesic flow when the ambient manifold is negatively curved. In this paper, following the recent work of Calegari-Marques-Neves, we study the asymptotic counting of surface subgroups in terms of areas of $k$-surfaces. We determine a lower bound, and we prove rigidity when this bound is achieved. Our work differs from that of Calegari-Marques-Neves in two key respects. Firstly, we work with all quasi-Fuchsian subgroups as opposed to merely asymptotically Fuchsian ones. Secondly, as their proof of rigidity breaks down in the present case, we require a different approach. Following ideas recently outlined by Labourie, we prove rigidity by solving a general foliated Plateau problem in Cartan-Hadamard manifolds. To this end, we build on Labourie's theory of $k$-surface dynamics, and propose a number of new constructions, conjectures and questions.

RÉSUMÉ. – Dans les années 2000, Labourie a commencé l'étude des propriétés dynamiques de l'espace des $k$-surfaces, c'est-à-dire des surfaces complètes à courbure extrinsèque constante dans les 3-variétés à courbure négative, qu'il présente comme des analogues du flot géodésique en dimension supérieure. Dans cet article, en suivant les travaux récents de Calegari-Marques-Neves, nous étudions le comptage asymptotique de sous-groupes de surfaces selon l'aire des $k$-surfaces qui les représentent. Nous établissons une borne inférieure, et prouvons un résultat de rigidité lorsque le minimum est atteint. Notre travail se distingue de celui de Calegari-Marques-Neves en deux aspects. Premièrement, nous considérons tous les sous-groupes quasi-fuchsiens et non pas uniquement ceux qui sont asymptotiquement fuchsiens. Deuxièmement, leur preuve de la rigidité ne s'applique pas dans notre cadre et nous suivons une autre approche. Nous établissons la rigidité en résolvant un problème de Plateau feuilleté général dans les variétés de Cartan-Hadamard. Pour ce faire, nous nous basons sur la théorie dynamique des k-surfaces développée par Labourie, et proposons quelques nouvelles constructions, conjectures et questions.

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## 1. Introduction

### 1.1. Asymptotic counting

In his pioneering work [22], Labourie initiated a dynamical systems' approach to the study of suitably complete surfaces of constant extrinsic curvature equal to $k > 0$ (henceforth referred to as $k$-*surfaces*). Indeed, let $X := (X, h)$ be an oriented 3-dimensional Cartan-Hadamard manifold of sectional curvature bounded above by $-1$. Suppose furthermore that $X$ is acted on cocompactly by the group $\Pi$, and that the quotient $X/\Pi$ is a manifold. Given $0 < k < 1$, Labourie introduced the space $\mathcal{S}_k(X/\Pi)$ of marked $k$-surfaces in $X/\Pi$. He showed how this space possesses a natural laminated structure, how it may be viewed as an extension of the geodesic lamination, and how it possesses the hyperbolic properties of the latter. In addition, he proposed the study of the asymptotic growth rate and possible rigidity properties as $E$ tends to infinity of the number $\mathrm{N}(E, X/\Pi)$ of compact elements of $\mathcal{S}_k(X/\Pi)$ of *energy* bounded above by $E$ (see Section 1.2 of [22] and Section 3.5.1 of [23]). This hard but intriguing problem remains open. In the present paper, inspired by the recent work [8] of Calegari-Marques-Neves, we address a simpler, but related, asymptotic counting problem expressed in terms of areas of compact elements of $\mathcal{S}_k(X/\Pi)$.

In order to state our result, we first note that

$$\partial_\infty X = \partial_\infty \Pi = \partial_\infty \mathbb{H}^3 = \hat{\mathbb{C}}. \tag{1.1}$$

For $k > 0$, we define a $k$-*disk* in $X$ to be an oriented, smoothly embedded surface $D \subseteq X$, of constant extrinsic curvature equal to $k$, whose closure $\overline{D} \subseteq X \cup \hat{\mathbb{C}}$ is homeomorphic to the closed unit disk in $\mathbb{R}^2$. Note that the boundary

$$\partial_\infty D := \overline{D} \setminus D \tag{1.2}$$

of any $k$-disk is an oriented Jordan curve in $\hat{\mathbb{C}}$. Conversely, in Theorem 2.1.1, we show that, for all $0 < k < 1$, every oriented Jordan curve $c$ in $\hat{\mathbb{C}}$ defines a unique $k$-disk $D := \mathrm{D}_{k,h}(c)$ in $X$.

Recall now that a compact surface subgroup $\Gamma \subseteq \Pi$ is said to be *quasi-Fuchsian* whenever its limit set $\partial_\infty \Gamma$ is a quasicircle. Let QF denote the set of conjugacy classes of quasi-Fuchsian subgroups of $\Pi$. For all $0 < k < 1$, we define the function $\mathrm{A}_{k,h} : \mathrm{QF} \to \mathbb{R}$ by

$$\mathrm{A}_{k,h}([\Gamma]) := \mathrm{Area}(\mathrm{D}_{k,h}(\partial_\infty \Gamma)/\Gamma). \tag{1.3}$$

THEOREM 1.1.1. – *For every metric $h$ over $X/\Pi$ of sectional curvature bounded above by $-1$,*

$$\liminf_{A\to\infty} \frac{\log\left(\#\{[\Gamma] \in \mathrm{QF} \mid \mathrm{A}_{k,h}(\Gamma) \leqslant A\}\right)}{A \log(A)} \geqslant \frac{(1-k)}{2\pi}, \tag{1.4}$$

*with equality holding if and only if $h$ is hyperbolic.*

In fact, we prove a stronger, more technical, result, namely Theorem 5.1.1, which is closer in spirit to that of Calegari-Marques-Neves, and from which Theorem 1.1.1 follows immediately.

Theorem 1.1.1 is best understood within the context of the asymptotic counting result proven by Kahn-Marković in [18], namely

$$\lim_{g\to\infty}\frac{\log\big(\#\{[\Gamma]\in \mathrm{QF}\mid \mathrm{g}(\Gamma)\leqslant g\}\big)}{2g\log(g)}=1, \tag{1.5}$$

where, for all $\Gamma \in \mathrm{QF}$, $\mathrm{g}(\Gamma)$ denotes its genus. It is worth noting that (1.5) is, in fact, the lesser of two asymptotic counting results proven by Kahn-Marković in that paper. Their second result counts, not conjugacy classes, but commensurability classes. Since commensurability classes do not distinguish covers of the same surface, whilst conjugacy classes do, a rigidity result analogous to Theorem 1.1.1 for the former would be of greater geometric interest. However, such a result lies beyond the scope of currently available techniques.

We view (1.5) as a *topological* asymptotic counting result, in the sense that it only concerns the structure of the fundamental group $\Pi$. In [8], Calegari-Marques-Neves make the key observation that minimal surfaces may be used to refine (1.5) to a *geometric* asymptotic counting result, that is, one which involves the structure of the metric $h$. In this manner, they obtain a rigidity result in the spirit of the volume entropy rigidity result [5] of Besson-Courtois-Gallot. Indeed, rigidity in Theorem 1.1 of [8] and Theorem 1.1.1 states that geometric asymptotic counting is greater than topological asymptotic counting, with equality holding if and only if the metric is hyperbolic. Note that Calegari-Marques-Neves' asymptotic counting result in fact differs from that of Kahn-Marković in that it involves a double limit—a byproduct of the various geometric challenges that arise when working with minimal surfaces. Since $k$-disks have simpler geometric properties, only a single limit is required in Theorem 1.1.1, allowing us to state a result closer in spirit to that of Kahn-Marković.

## 1.2. Foliated Plateau problems

Theorem 1.1.1 differs substantially from the corresponding result of Calegari-Marques-Neves by the fact that its proof requires an extensive study of the dynamical properties of the space of $k$-disks. In the spirit of Gromov's theory of foliated Plateau problems—laid out in [12] and [13] and applied to $k$-surfaces by Labourie in [22] and [23]—we are led to view $k$-disks "not as individuals, but as members of a community," and then to investigate the dynamical properties of such families. We find the resulting theory to be interesting in its own right, and we thus also discuss certain new constructions and open problems which lie beyond our immediate needs for the study of asymptotic counting.

We describe here two successive refinements of Labourie's laminated structure obtained upon restricting to successively smaller subsets of $\mathcal{S}_k(X/\Pi)$. For ease of presentation, we only address the case where $X/\Pi$ is compact. More general statements of the following results are given in Sections 2 and 3.

Our first refinement yields a space of marked $k$-surfaces carrying a natural *fibered* structure. For all $0<k<1$, we define a *marked $k$-disk* in $X$ to be a pair $(D,p)$, where $D$ is a $k$-disk in $X$ and $p$ is a point of $D$. For all $0<k<1$, and for all $K\geqslant 1$, let $\mathrm{MKD}_{k,h}(K)$ denote the space of marked $k$-disks $(D,p)$ in $X$ whose ideal boundary $\partial_\infty D$ is a $K$-quasicircle in $\hat{\mathbb{C}}$,