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Tatsuro KAWAKAMI, Teppei TAKAMATSU, Hiromu TANAKA, Jakub WITASZEK, Fuetaro YOBUKO & Shou YOSHIKAWA

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Annales Scientifiques de l'École Normale Supérieure,
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Tél. : (33) 1 44 32 20 88.
Email : annales@ens.fr

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Société Mathématique de France
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Tél. : (33) 04 91 26 74 64
Email : abonnements@smf.emath.fr

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QUASI-F-SPLITTINGS IN BIRATIONAL GEOMETRY

BY Tatsuro KAWAKAMI, Teppei TAKAMATSU,
Hiromu TANAKA, Jakub WITASZEK, Fuetaro YOBUKO
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ABSTRACT. – We develop the theory of quasi-F-splittings in the context of birational geometry. Amongst other things, we obtain results on liftability of sections and establish a criterion for whether a scheme is quasi-F-split employing the higher Cartier operator. As one of the applications of our theory, we prove that three-dimensional klt singularities in large characteristic are quasi-F-split, and so, in particular, they lift modulo p^2.

RÉSUMÉ. – Nous développons la théorie des quasi-F-scindages dans le contexte de la géométrie birationnelle. Entre autres, nous obtenons des résultats sur le relèvement des sections et établissons un critère pour déterminer si un schéma est quasi-F-scindé en utilisant l'opérateur de Cartier supérieur. Comme application de notre théorie, nous prouvons que les singularités klt tridimensionnelles en grande caractéristique sont quasi-F-scindées et, en particulier, qu'elles se relèvent modulo p^2.

1. Introduction

What allowed for many recent developments in both commutative algebra and birational geometry (see, for example, [22]) was the study of F-*split* varieties, that is, varieties X in positive characteristic for which the Frobenius homomorphism $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ splits. The importance of this notion stems from the fact that it interpolates between arithmetic and geometric properties of algebraic varieties. Specifically, projective F-split varieties are always *weakly ordinary* (the Frobenius acts bijectively on $H^{\dim X}(X, \mathcal{O}_X)$) and they have non-positive Kodaira dimension. In the case of Calabi-Yau varieties, being F-split and ordinary is equivalent.

Unfortunately, F-splittings are irrelevant to the study of non-weakly ordinary varieties, even though such varieties may still carry interesting arithmetic information. Namely, every Calabi-Yau variety X can be assigned its *Artin-Mazur height* $\mathrm{ht}(X) \in \mathbb{Z}_{>0} \cup \{\infty\}$ which is a crystalline-type invariant measuring supersingularity. Having height one is equivalent to ordinarity, whilst having finite height may be thought of as mild non-ordinarity.

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Elliptic curves can only have height 1 (ordinary) or height 2 (supersingular), but the situation is more complicated in higher dimensions. For example, if X is a K3 surface, then $\mathrm{ht}(X) \in \{1, \dots, 10\} \cup \{\infty\}$ and every such height occurs. In view of the above, it is natural to wonder if one could find a similar notion to an F-splitting in the context of finite Artin-Mazur heights, and indeed, such a notion was found and introduced by the fifth author in [66]. One calls a variety *n-quasi-F-split* if there exists a dashed arrow rendering the following diagram

$$\begin{array}{ccc} W_n\mathcal{O}_X & \xrightarrow{\ F\ } & F_*W_n\mathcal{O}_X \\ \big\downarrow{\scriptstyle R^{n-1}} & \swarrow{\scriptstyle \exists} & \\ \mathcal{O}_X & & \end{array} \tag{1.0.1}$$

in the category of $W_n\mathcal{O}_X$-modules commutative, where $W_n\mathcal{O}_X$ is a sheaf of Witt vectors. We will say that X is *quasi-F-split* if it is n-quasi-F-split for some $n \in \mathbb{Z}_{>0}$. In analogy to F-splittings, if X is a Calabi-Yau variety, then the Artin-Mazur height $\mathrm{ht}(X)$ is equal to the smallest integer $n \in \mathbb{Z}_{>0}$ for which X is n-quasi-F-split. What is striking about quasi-F-splittings is that, despite them being much less restrictive, they share many properties with F-splittings. For instance, quasi-F-split varieties lift modulo p^2 [66, 1] and satisfy Kodaira vanishing [47].

The main objective of our article is to generalize the theory of F-splittings to quasi-F-splittings in the context of birational geometry. In particular, as one of the applications, we show the following.

THEOREM A (Theorem 6.19). – *There exists a positive integer $p_0 > 0$ such that the following holds. Let X be a three-dimensional $\mathbb{Q}$-factorial affine klt variety over a perfect field of characteristic $p \geq p_0$. Then X is quasi-F-split. In particular, it lifts modulo p^2.*

Moreover, assuming a natural generalization of the work of Arvidsson-Bernasconi-Lacini on log liftability of log del Pezzo pairs with standard coefficients (Conjecture 6.20), we show that $p_0 = 43$ works in the above theorem (see Remark 6.21). Note that Theorem A is false for F-splittings as shown in [10, Theorem 1.1]. We also point out that it is usually very hard to prove liftability of singularities as their deformation theory is governed by the cotangent complex which is rarely computable in practice.

We introduce two techniques for verifying whether a variety is quasi-F-split: inversion of adjunction and a criterion employing the higher Cartier operator. By using inversion of adjunction and the local-to-global correspondence between klt singularities and log Fano varieties, one can reduce Theorem A to the study of whether log del Pezzo surfaces are quasi-F-split. In this context, we can show the following.

THEOREM B (Corollary 6.4, cf. Theorem 6.18). – *Let X be a klt del Pezzo surface over a perfect field of characteristic $p > 5$. Then X is quasi-F-split.*

In fact, we prove that if (X, Δ) is a log del Pezzo pair in characteristic $p \gg 0$ with Δ having standard coefficients (that is of the form $1 - \frac{1}{n}$ for $n \in \mathbb{Z}_{>0}$), then (X, Δ) is 2-quasi-F-split (Theorem 6.18). These results are based on [9] and [2].

One of the foundational results in the theory of F-splittings is the theorem of Hara [24], who proved that affine klt surfaces are F-split in characteristic $p > 5$. We show a stronger result for quasi-F-splittings.

THEOREM C (Corollary 5.17). – *Let X be an affine klt surface over a perfect field of characteristic $p > 0$. Then X is quasi-F-split.*

In fact, we prove that every two-dimensional affine klt pair (X, Δ) in every positive characteristic is quasi-F-split. Surprisingly, one need not even assume that Δ has standard coefficients.

REMARK 1.1. – In Theorems A, B, and C, the choice of splittings is not natural amongst other varieties. Indeed, these theorems depend on Theorem F (cf. Theorem 5.13), which ensures only the existence of splittings.

Before moving on to discussing the aforementioned two key concepts (Subsection 1.1 and Subsection 1.2), we point out that multiple other foundational results on quasi-F-splittings are contained in this article, for example, invariance under birational and finite maps (Corollary 3.44 and Corollary 3.20), non-positivity of Kodaira dimension (Proposition 3.14), and Kawamata-Viehweg vanishing for quasi-F-split pairs (Theorem 3.15). Furthermore, one of the starting problems of our work is finding a correct definition of a quasi-F-splitting for log pairs (X, Δ). To this end, we build on the construction, due to the third author, of Witt vectors with $\mathbb{Q}$-boundary (see [63]).

1.1. Lifting sections and inversion of adjunction

Inversion of adjunction is a key tool in the study of F-splittings. Briefly speaking, in the local setting, it says that if X is a normal variety and $S \subseteq X$ is an F-pure normal prime divisor such that $K_X + S$ is Cartier, then X is F-pure along S [51, main Theorem]. In the global setting, it stipulates that if X is projective over a field of positive characteristic, $-(K_X + S)$ is ample, and S is F-split, then X is F-split, too.

Unfortunately, local inversion of adjunction is false for quasi-F-splittings (see [35, Example 7.7]). What is true however is that global quasi-F-split inversion of adjunction holds when $-S$ is semiample. Typical situations when this is the case is when X admits a projective morphism $\pi : X \to Z$ such that either π is birational and $-S$ is ample, or such that Z is a curve and S is a fiber of π.

THEOREM D (Corollary 4.12). – *Let X be a smooth variety admitting a projective morphism $\pi : X \to Z$ to an affine normal variety Z over a perfect field of characteristic $p > 0$. Let S be a smooth prime divisor on X. Assume that S is quasi-F-split, $-S$ is semiample, $A := -(K_X + S)$ is ample, and*

$$H^1(X, \mathcal{O}_X(K_X + p^i A - kS)) = 0$$

for every $i \geq 1$ and $k \geq 0$. Then X is quasi-F-split over an open neighborhood of $\pi(S)$.