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ON AFFINE LUSZTIG VARIETIES

BY XUHUA HE

ABSTRACT. – Affine Lusztig varieties encode the orbital integrals of Iwahori-Hecke functions and serve as building blocks for the (conjectural) theory of affine character sheaves. We establish a close relationship between affine Lusztig varieties and affine Deligne-Lusztig varieties. Consequently, we give an explicit nonemptiness pattern and dimension formula for affine Lusztig varieties in most cases.

RÉSUMÉ. – Les variétés de Lusztig affines codent les intégrales orbitales des fonctions d'Iwahori-Hecke et servent d'éléments de base pour la théorie (conjecturale) des faisceaux caractères affines. Nous établissons une relation étroite entre les variétés de Lusztig affines et les variétés de Deligne-Lusztig. En conséquence, nous donnons un modèle explicite de non-vacuité et une formule de dimension pour les variétés de Lusztig affines dans la plupart des cas.

1. Introduction

1.1. Deligne-Lusztig varieties and Lusztig varieties

Let $\mathbf{H}$ be a connected reductive group over a finite field $\mathbb{F}_q$. Let $H = \mathbf{H}(\bar{\mathbb{F}}_q)$ and Fr be the Frobenius endomorphism on H. Let B be an F-stable Borel subgroup of H and W be the Weyl group of H. The Deligne-Lusztig variety is defined by

$$X_w = \{hB; h^{-1} \operatorname{Fr}(h) \in B\dot{w}B\},$$

where $w \in W$ and $\dot{w}$ is a representative of w in G. It is the subvariety of the flag variety H/B consisting of points with fixed relative position with its image under the Frobenius endomorphism. Deligne-Lusztig varieties play a crucial role in the representation theory of finite groups of Lie type [10].

Lusztig varieties were introduced in 1977 by Lusztig [35] at the *LMS symposium on representations of Lie groups* in Oxford. They are defined by replacing the Frobenius endomorphism in the definition of X_w with the conjugation action by a given regular semisimple element $h \in H$. It was shown in [35] that the character values arising from the cohomology of Deligne-Lusztig varieties are "universal invariants," which also make sense over complex numbers. The definition of Lusztig varieties was generalized to an arbitrary element $h \in H$ in Lusztig's theory of character sheaves [36]. Character sheaves can be defined in two different ways: via "admissible complexes" and via Lusztig varieties. It is a deep result that the two definitions coincide. Abreu and Nigro [1] used Lusztig varieties in their geometric approach to characters of Hecke algebras. The homology classes of Deligne-Lusztig varieties and those of Lusztig varieties were studied by Kim [29].

1.2. Affine Lusztig varieties

The main subjects of this paper are the affine analogs of Lusztig varieties.

Let $\mathbf{k}$ be an algebraically closed field and $L = \mathbf{k}((\varepsilon))$ be the field of the Laurent series (equal characteristic case) or $W(\mathbf{k})[1/p]$ (mixed characteristic case) if $\mathbf{k}$ is characteristic p, where $W(\mathbf{k})$ is the ring of p-typical Witt vectors. Let $\mathbf{G}$ be a connected reductive group over L. Let $\breve{G} = \mathbf{G}(L)$. Let $\mathrm{Gr} = \breve{G}/\mathcal{K}^{\mathrm{sp}}$ be the affine Grassmannian of $\breve{G}$ and $\mathrm{Fl} = \breve{G}/\mathcal{I}$ be the affine flag variety. We define the affine Lusztig varieties

$$Y_\mu^{\mathbf{G}}(\gamma) = \{g\mathcal{K}^{\mathrm{sp}}; g^{-1}\gamma g \in \mathcal{K}^{\mathrm{sp}} t^\mu \mathcal{K}^{\mathrm{sp}}\},$$
$$Y_w^{\mathbf{G}}(\gamma) = \{g\mathcal{I} \in \mathrm{Fl}; g^{-1}\gamma g \in \mathcal{I}\dot{w}\mathcal{I}\}$$

in the affine Grassmannian and the affine flag variety, respectively. Here γ is a regular semisimple element in $\breve{G}$, μ is a dominant coweight of $\breve{G}$, and w is an element in the Iwahori-Weyl group of $\breve{G}$. Lusztig first introduced affine Lusztig varieties in [39]. Affine Lusztig varieties in the affine Grassmannian were also studied by Kottwitz and Viehmann [30]. In the literature, affine Lusztig varieties are also called generalized affine Springer fibers or Kottwitz-Viehmann varieties.

In equal characteristic setting, affine Lusztig varieties are locally closed subschemes of the affine Grassmannian and affine flag variety, equipped with a reduced scheme structure. In mixed characteristic setting, we consider affine Lusztig varieties as perfect schemes in the sense of Zhu [50] and Bhatt-Scholze [4], i.e., as locally closed perfect subschemes of p-adic flag varieties.

Affine Lusztig varieties in the affine Grassmannian and the affine flag variety arise naturally in the study of orbital integrals of spherical Hecke algebras and Iwahori-Hecke algebras. Roughly speaking, based on the Grothendieck-Lefschetz trace formula, one may interpret the orbital integrals of spherical Hecke algebras (resp. Iwahori-Hecke algebras) as point-counting problems on the associated affine Lusztig varieties in the affine Grassmannian (resp. the affine flag variety). We refer to Yun's lecture notes [49], and recent preprint of Chi [9, §5.3] for some formulas connecting the affine Springer fibers and orbital integrals and the thesis work of G. Wang [48, Proposition 11.2.5] for some formulas connecting the affine Lusztig varieties in the affine Grassmannian and the orbital integrals.

Affine Lusztig varieties also serve as building blocks for the (conjectural) theory of affine character sheaves. Namely, we consider the map

$$m : \breve{G} \times^{\mathcal{I}} \mathcal{I}\dot{w}\mathcal{I} \longrightarrow \breve{G}, \qquad (g, g') \longmapsto gg'g^{-1},$$

where $\breve{G} \times^{\mathcal{I}} \mathcal{I}\dot{w}\mathcal{I}$ is the quotient of $\breve{G} \times \mathcal{I}\dot{w}\mathcal{I}$ by the $\mathcal{I}$-action defined by $i \cdot (g, g') = (gi^{-1}, ig'i^{-1})$. The fibers of this map are the affine Lusztig varieties in the affine flag variety. This map is the affine analog of Lusztig's map to define the character sheaves for reductive groups over algebraically closed fields. One may define a similar map by replacing $\mathcal{I}$ with $\mathcal{K}^{\mathrm{sp}}$. It is interesting to realize Lusztig's unipotent almost characters of p-adic groups [40] using affine Lusztig varieties.

1.3. Main result

It is a fundamental question to determine the nonemptiness pattern and dimension formula for affine Lusztig varieties.

This question was solved for affine Lusztig varieties in the affine Grassmannian of a split connected reductive group in equal characteristic case (under a mild assumption on the residue characteristic) by Bouthier and Chi [5, 6, 8]. Their method relies on a global argument using the Hitchin fibration and does not work in mixed characteristic case (e.g., for $Y_\mu(\gamma)$ for a ramified element γ). See [8, Remark 1.2.3]. Little is known for nonsplit groups or for the affine flag case.

The main purpose of this paper is to study affine Lusztig varieties in the affine flag variety of any (not necessarily split) connected reductive group in equal and mixed characteristic cases. Our approach is very different from previous approaches. The key idea is to establish a close relationship between affine Lusztig varieties and their Frobenius-twisted analogs, affine Deligne-Lusztig varieties. Our approach is purely local and thus works also in mixed characteristic case. We will focus on the affine flag case, since the affine Grassmannian case can be deduced from this.

We first give a quick review of affine Deligne-Lusztig varieties. Let F' be a nonarchimedean local field and $\breve{F}'$ the completion of the maximal unramified extension of F'. Let $\mathbf{G}'$ be a connected reductive group over F', $\breve{G}' = \mathbf{G}'(\breve{F}')$ and σ be the Frobenius endomorphism on $\breve{G}'$. Let $\mathcal{I}'$ be a σ-stable Iwahori subgroup of $\breve{G}'$. Let $b \in \breve{G}'$ and w be an element in the Iwahori-Weyl group of $\breve{G}'$. The affine Deligne-Lusztig variety $X_w(b)$ is defined by

$$X_w^{\mathbf{G}'}(b) = \{g\mathcal{I}' \in \mathrm{Fl}'; g^{-1}b\sigma(g) \in \mathcal{I}'\dot{w}\mathcal{I}'\}.$$

The notion of an affine Deligne-Lusztig variety was first introduced by Rapoport [43]. It plays an important role in arithmetic geometry and the Langlands program. Affine Deligne-Lusztig varieties have been studied extensively in the past two decades, and we now have a much better understanding of these varieties than of affine Lusztig varieties. The nonemptiness pattern and the dimension formula for affine Deligne-Lusztig varieties are completely known in the affine Grassmannian case and are known for most pairs (w, b) in the affine flag case. We refer to the survey article [21] for a detailed discussion.

The structure of affine Deligne-Lusztig varieties, in general, is quite complicated, and affine Lusztig varieties are even more complicated, partially because, in the loop group, the classification of ordinary conjugacy classes is more complicated than the classification of Frobenius-twisted conjugacy classes. In the special case where $w = 1$, the associated affine Deligne-Lusztig variety is either empty or a discrete set. The associated affine Lusztig variety is the affine Springer fiber Fl^γ and is, in general, finite-dimensional (but not zero-dimensional). The dimension formula for affine Springer fiber was obtained by Bezrukavnikov [2] in equal characteristic case and Chi [9] in mixed characteristic case.

The main result of this paper, roughly speaking, shows that the difference between affine Lusztig varieties and affine Deligne-Lusztig varieties comes from the affine Springer fibers.