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CONSTRUCTION OF HIGHER-DIMENSIONAL ALF CALABI-YAU METRICS

BY Daheng MIN

Abstract. – Roughly speaking, an ALF metric of real dimension $4n$ should be a metric such that it has a $(4n-1)$-dimensional asymptotic cone, the volume growth of this metric is of order $4n-1$ and its sectional curvature tends to 0 at infinity.

In this paper, we first show that the Taub-NUT deformation of a hyperkähler cone with respect to a locally free $\mathbb{S}^1$-symmetry is ALF hyperkähler. Using this metric at infinity, we establish the existence of ALF Calabi-Yau metric on certain crepant resolutions. In particular, we prove that there exist ALF Calabi-Yau metrics on canonical bundles of classical homogeneous Fano contact manifolds.

Résumé. – Informellement, une métrique ALF de dimension réelle $4n$ devrait être une métrique telle que son cône asymptotique est de dimension $4n-1$, la croissance de volume de cette métrique est d'ordre $4n-1$ et sa courbure sectionnelle tend vers 0 à l'infini.

Dans cet article, on montre d'abord que la déformation de Taub-NUT d'un cône hyperkählérien par rapport à une symétrie de $\mathbb{S}^1$ localement libre est une métrique ALF hyperkählérienne. En utilisant cette métrique à l'infini, on établit l'existence d'une métrique ALF de Calabi-Yau sur certaines résolutions crépantes. En particulier, on démontre qu'il existe des métriques ALF de Calabi-Yau sur les fibrés canoniques des variétés de contact de Fano homogènes classiques.

1. Introduction

1.1. Motivation

After Yau's celebrated confirmation of Calabi conjecture [47] which implies the existence of a Ricci-flat Kähler metric on a compact Kähler manifold satisfying certain conditions, there have been many works on non-compact Calabi-Yau metrics. Among those works, a particular class introduced by Hawking [26], namely gravitational instanton, is of special interest. They are complete hyperkähler 4-manifolds with a decaying curvature at infinity.

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Under the assumption of finite energy, the gravitational instantons are classified into four types according to their 'dimension m at infinity' (see [41, Section 6.4] for the precise definition of the following spaces). We have ALE ($m = 4$), ALF ($m = 3$), ALG and ALG* ($m = 2$), ALH and ALH* ($m = 1$) gravitational instantons. (See also [11] for a classification under the assumption of a faster than quadratic curvature decay. In this case, it means that the metric is locally asymptotic to a $\mathbb{T}^{4-m}$-fibration over $\mathbb{R}^m$ near the infinity).

The gravitational instantons have been very well understood. Examples of ALE gravitational instantons are found by Eguchi and Hanson [21], Gibbons and Hawking [23] and Hitchin [30]. A complete classification of ALE gravitational instantons is given by Kronheimer [32, 33], showing that ALE gravitational instantons are crepant resolutions of $\mathbb{C}^2/G$ where G is a finite subgroup of $\mathrm{SU}(2)$. For ALF gravitational instantons, examples of cyclic type, known as multi-Taub-NUT metrics, are constructed by Hawking [26], while Cherkis and Kapustin [13, 14] find the dihedral type. The cyclic type is classified by Minerbe [38] and then a complete classification of ALF instantons is given by Chen and Chen [10].

Certain aspects of ALE gravitational instantons can be generalized to higher dimensions. Bando, Kasue and Nakajima [3] show that faster than quadratic curvature decay and Euclidean volume growth imply ALE property. Joyce [31] systematically studies certain resolutions of $\mathbb{C}^m/G$ where G is a finite subgroup of (m) that may not act freely on $\mathbb{C}^m \setminus \{0\}$. More generally, if we allow for the quadratic decay of curvature, or arbitrary asymptotic cone, then asymptotically conic Calabi-Yau manifolds can be thought as a generalization of ALE instantons to higher dimensions. To this end, in addition to Joyce's works, we have works by Goto [25] and Van Coevering [15] which study the crepant resolution of a Calabi-Yau cone. We also have works by Conlon and Hein [17] which study both resolutions and deformations of Calabi-Yau cones.

Compared to ALE instantons, less is known about how to generalize the theory of ALF instantons to higher dimensions. Chen and Li [12] show that under a faster than quadratic curvature decay and some holonomy control, the metric is asymptotically a torus fibration over an ALE space. However, there is no known example of a higher-dimensional ALF Calabi-Yau manifold satisfying their assumptions.

In analogy to ALF instantons, we propose the following notion of a higher-dimensional ALF metric. An ALF metric of real dimension $4n$ should be a complete metric such that it has a $(4n-1)$-dimensional asymptotic cone, the volume growth of this metric is of the order $4n-1$ and its sectional curvature tends to 0 at infinity. In this article, we give an affirmative response to the following question: Does there exist an ALF Calabi-Yau metric of dimension strictly larger than 4?

1.2. Statement of results

The first main result of this article is as follows.

THEOREM 1.1. – *Let S be a compact connected 3-Sasakian manifold of dimension* $4n-1$ *($n \geq 1$) admitting a locally free $\mathbb{S}^1$-symmetry. Let (M, g_0, I_i, ω_i) be the hyperkähler cone over S, then for any $a > 0$, a certain deformation (known as the Taub-NUT deformation) $(M, g_a, I_i^a, \omega_i^a)$ with respect to this $\mathbb{S}^1$-symmetry is an ALF hyperkähler metric on M in the sense that the sectional curvature K_{g_a} is bounded by $\frac{C}{\rho_a}$ for some $C > 0$, the volume growth* *of* g_a *is of order* $4n-1$ *and its asymptotic cone is* $M_0 \times \mathbb{R}^3$*, which is a metric cone of dimension* $4n-1$*. Here* ρ_a *is the distance function measured by the deformed metric* g_a*, and* M_0 *is the hyperkähler quotient of* M *with respect to the* $\mathbb{S}^1$*-symmetry and hyperkähler moment* $0 \in \mathbb{R}^3$.

We will explain the Taub-NUT deformation in Section 2, here we point out that in the special case of $S = \mathbb{S}^3$, the resulting metric in Theorem 1.1 is the Taub-NUT metric on $\mathbb{C}^2$, which is known to be ALF. The passage from the Euclidean metric on $\mathbb{C}^2$ to the Taub-NUT metric is known as the Gibbons-Hawking deformation, and the Taub-NUT deformation is a generalization of the Gibbons-Hawking deformation to higher dimensions. In this way, we think of Theorem 1.1 as a generalization of the Taub-NUT metric. A special case of $S = \mathbb{S}^{4n-1}$ of the theorem produces the so-called Taubian-Calabi metric; in this case M_0 is a toric hyperkähler cone depending on the choice of the $\mathbb{S}^1$-symmetry. According to [24], the name Taubian-Calabi is due to Roček [39]. It follows that Theorem 1.1 implies that the Taubian-Calabi metric has an ALF property. Besides the case $S = \mathbb{S}^{4n-1}$, we also give other applications of Theorem 1.1 in Section 7.

One feature of Theorem 1.1 is that it allows us to produce many ALF hyperkähler metrics on the same complex manifold. As an example, we will show that, in addition to the Taubian-Calabi metric, there exist infinitely many ALF hyperkähler metrics on $\mathbb{C}^{2n}$ with different asymptotic cones (see Example 7.4). This phenomenon also appears in the context of asymptotically conic Calabi-Yau metric, we mention the works [37], [19] and [42] which give counterexamples to a conjecture of Tian [44, Remark 5.3].

Another feature of Theorem 1.1 is that the asymptotic cone $M_0 \times \mathbb{R}^3$ is generally not a smooth cone. For example, $\{0\} \times \mathbb{R}^3$ may be a singular locus. We will see in Section 4 and Section 5 how this singularity in the asymptotic cone is resolved in the ALF metric. Examples of asymptotically conic Calabi-Yau metrics with singular asymptotic cone are studied by Joyce [31] (QALE manifold), Székelyhidi [42], Yang Li [37], Conlon, Degeratu and Rochon [16, 19].

The proof of Theorem 1.1 is based on a careful study of the Taub-NUT deformation from Section 2 to Section 5.

Observe that as long as the 3-Sasakian manifold is not isometrically the unit sphere, its cone M is not smooth at the vertex. Thus, we are led to consider resolution of quotients of M. This is the motivation of the second main result:

THEOREM 1.2. – *With the assumptions of Theorem 1.1, suppose furthermore that* $n \geq 2$ *and there is a finite group* Γ *acting on* S *whose extension to* M *preserves one of the complex structures* I_1 *and its deformed Kähler potential* K^a_{ALF}*. Let* $\pi : Y \to M/\Gamma$ *be an* I_1*-holomorphic crepant resolution, then for any compactly supported Kähler class of* Y *and any* $c > 0$*, there exists an ALF Calabi-Yau metric* ω *in this class which is asymptotic to* $c\omega^a_1$ *near the infinity. More precisely, we have*

$$|\nabla^k(\omega - c\pi^*(i\partial\bar{\partial}K^a_{\mathrm{ALF}}))|_\omega \leq C(k,\varepsilon)(1+\rho_\omega)^{-4n+3+\varepsilon},$$

where $\varepsilon > 0$ *is sufficiently small,* ρ_ω *is the distance from a point in* Y *measured by* ω *and* $k \geq 0$*. Here* K^a_{ALF} *is invariant by* Γ *so we think of it as a function on* M/Γ.