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HAUSDORFF DIMENSION AND EXACT APPROXIMATION ORDER IN $\mathbb{R}^n$

BY PRASUNA BANDI AND NICOLAS DE SAXCÉ

ABSTRACT. – Given a non-increasing function $\psi:\mathbb{N}\to\mathbb{R}^+$ such that $s^{\frac{n+1}{n}}\psi(s)$ tends to zero as s goes to infinity, we show that the set of points in $\mathbb{R}^n$ that are exactly ψ-approximable is non-empty, and we compute its Hausdorff dimension. This answers questions of Jarník and of Beresnevich, Dickinson and Velani.

RÉSUMÉ. – Pour toute fonction décroissante $\psi:\mathbb{N}\to\mathbb{R}^+$ telle que $s^{\frac{n+1}{n}}\psi(s)$ tend vers zéro lorsque s tend vers l'infini, nous montrons que l'ensemble des points dans $\mathbb{R}^n$ qui sont approchables exactement à l'ordre ψ est non vide, et nous calculons sa dimension de Hausdorff. Cela répond à des questions de Jarník et de Beresnevich, Dickinson et Velani.

1. Introduction

Any rational point v in $\mathbb{R}^n$ can be written uniquely as $v=\frac{p}{q}$, with p an integer vector in $\mathbb{Z}^n$ and q a positive integer such that the $(n+1)$-tuple (p,q) is primitive. With this notation, given a non-increasing function $\psi:\mathbb{N}\to\mathbb{R}_+^*$, one defines the set of ψ-approximable points in $\mathbb{R}^n$ as

$$W_n(\psi)=\left\{x\in\mathbb{R}^n \mid \text{there exist infinitely many } \frac{p}{q}\in\mathbb{Q}^n \text{ with } \left\|x-\frac{p}{q}\right\|<\psi(q)\right\},$$

where the norm on $\mathbb{R}^n$ is given by $\|x\|=\max_{1\le i\le n}|x_i|$ if $x=(x_1,\dots,x_n)$. For $\tau>0$, let

$$\psi_\tau(s)=s^{-\tau}.$$

It follows from Dirichlet's celebrated theorem that $W_n(\psi_{\frac{n+1}{n}})=\mathbb{R}^n\setminus\mathbb{Q}^n$. For larger values of τ, Jarník [14] obtained the following theorem.

THEOREM 1 (Jarník, 1931). – *Let $n\ge 1$ be an integer. For every $\tau\ge\frac{n+1}{n}$, the set $W_n(\psi_\tau)$ has Hausdorff dimension*

$$\dim_H W_n(\psi_\tau)=\frac{n+1}{\tau}.$$

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Since the expression $\frac{n+1}{\tau}$ is decreasing as a function of τ, Jarník's formula shows in particular that if $\tau' > \tau$, then the set $W_n(\psi_{\tau'})$ is strictly smaller than $W_n(\psi_\tau)$. On the other hand, it does not allow one to distinguish the two sets $W_n(\psi_\tau)$ and $W_n(\frac{1}{2}\psi_\tau)$. However, for τ large enough Jarník was able to sharpen his result, as he observed that for every $\tau > 2$ and every $c < 1$, one even has a strict inclusion

$$W_n(c\psi_\tau) \subsetneq W_n(\psi_\tau).$$

For $n > 1$, the condition $\tau > 2$ is unnatural, and Jarník's remarks at the end of his paper suggest that the result should hold for any $\tau > \frac{n+1}{n}$. One goal of this paper is to show that this is indeed the case.

More precisely, defining the set of *exact ψ-approximable* vectors in $\mathbb{R}^n$ by

$$E_n(\psi) = W_n(\psi) \setminus \bigcup_{c<1} W_n(c\psi),$$

we shall prove the following result generalizing Jarník [14, Satz 6].

Theorem 2 (Existence of exact ψ-approximable vectors). – *Let $n \geq 1$ be an integer. If $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ is non-increasing and satisfies $\lim_{s\to\infty} s^{\frac{n+1}{n}}\psi(s) = 0$, then $E_n(\psi) \neq \emptyset$.*

In the particular case $n = 1$, Bugeaud [4, 5] and then Bugeaud and Moreira [6] studied the sets $E_1(\psi)$ from the point of view of Hausdorff dimension, and showed that $\dim_{\mathrm{H}} E_1(\psi) = \dim_{\mathrm{H}} W_1(\psi)$ provided $s^2\psi(s)$ tends to zero at infinity. Under the assumption that $\lim_{s\to\infty} -\frac{\log\psi(x)}{\log(x)}$ exists and is at least 2, Fregoli [13] was able to compute the Hausdorff dimension of $E_n(\psi)$ in the case $n \geq 3$ but as Jarník himself already observed, the condition $\psi(s) = o(s^{-2})$ is too restrictive when $n \geq 2$, and should be replaced by $\psi(s) = o(s^{-\frac{n+1}{n}})$. In [1] Bandi, Ghosh, and Nandi studied the exact approximation problem in the abstract set-up of Ahlfors regular metric spaces but again, their assumptions imply in particular that the abstract rational points satisfy a certain well-separatedness property, which the rationals in $\mathbb{R}^n$ do not satisfy for $n \geq 2$. A variant of the problem was studied by Beresnevich, Dickinson, and Velani [2] who showed that

$$\dim_{\mathrm{H}}(W_n(\psi_1) \setminus W_n(\psi_2)) = \dim_{\mathrm{H}} W_n(\psi_1)$$

under certain assumptions that imply in particular that $\frac{\psi_1(s)}{\psi_2(s)}$ tends to infinity as s goes to infinity. They observed however that their techniques completely fail if one takes $\psi_2 = c\psi_1$, and that new ideas and methods would be needed to cover this case. Our approach allows us to give a satisfactory answer to this problem, by showing that Bugeaud's result is in fact valid in any dimension. The next theorem is the main result of our paper.

Theorem 3 (Hausdorff dimension of exact approximable vectors).

Let $n \geq 1$ be an integer. Assume that $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ is non-increasing with $\psi(s) = o(s^{-\frac{n+1}{n}})$. Then the set of exact ψ-approximable vectors in $\mathbb{R}^n$ satisfies

$$\dim_{\mathrm{H}} E_n(\psi) = \dim_{\mathrm{H}} W_n(\psi).$$

In [2], the authors also define the set of ψ-badly approximable points

$$\mathbf{Bad}_n(\psi) = W_n(\psi) \setminus \bigcap_{c>0} W_n(c\psi)$$

and suggest to study the Hausdorff dimension of this set. We obtain a complete answer to that problem as an immediate corollary of Theorem 3.

COROLLARY 1.1 (Hausdorff dimension of ψ-badly approximable points).
Let $\psi\colon \mathbb{N} \to \mathbb{R}_+^*$ be a non-increasing function such that $\psi(s) = o(s^{-\frac{n+1}{n}})$. Then

$$\dim_{\mathrm{H}} \mathbf{Bad}_n(\psi) = \dim W_n(\psi).$$

We note however that this corollary can be obtained more easily using the *variational principle in the parametric geometry of numbers* of Das, Fishman, Simmons and Urbański [7]. This alternative argument is sketched in Paragraphs 2.2 and 2.3, as an introduction to the tools and techniques that will be further developed for the proof of Theorem 3. In the particular case of $\psi(s) = s^{-\lambda}$, Corollary 1.1 was obtained independently by Koivusalo, Levesley, Ward and Zhang [17] using different methods.

Theorem 3 above is new for $n \geq 2$ even in the case of the elementary function $\psi_\tau(s) = s^{-\tau}$ for $\tau > \frac{n+1}{n}$. In that case, the formula for the Hausdorff dimension is particularly simple: The set $E_n(\psi_\tau)$ of points x in $\mathbb{R}^n$ for which there exist infinitely many rational points $\frac{p}{q}$ such that $\left\|x - \frac{p}{q}\right\| < q^{-\tau}$, but only finitely many satisfying $\left\|x - \frac{p}{q}\right\| < cq^{\tau}$ if $c < 1$, satisfies

$$\dim_{\mathrm{H}} E_n(\psi_\tau) = \frac{n+1}{\tau}.$$

More generally, one defines the *lower order at infinity* of ψ, denoted λ_ψ, to be

$$\lambda_\psi := \liminf_{s\to\infty} \frac{-\log \psi(s)}{\log s}.$$

A result of Dodson [8] shows that if $\lambda_\psi \geq \frac{n+1}{n}$, the dimension of $W_n(\psi)$ is given by

$$\dim_{\mathrm{H}} W_n(\psi) = \frac{n+1}{\lambda_\psi}.$$

In Theorem 3, only the lower bound $\dim_{\mathrm{H}} E_n(\psi) \geq \dim_{\mathrm{H}} W_n(\psi)$ requires a proof, and for that we shall construct inside $E_n(\psi)$ a Cantor set with the required Hausdorff dimension $\frac{n+1}{\lambda_\psi}$.

To construct that Cantor set, the general strategy is similar to the one developed by Bugeaud in [4], using balls of the form $B(y_k, \frac{\psi(H(v_k))}{k})$, where v_k is a rational point and y_k is chosen so that $d(y_k, v_k) = (1 - \frac{1}{k})\psi(H(v_k))$. It is clear that any point x lying in infinitely many such balls is ψ-approximable, but not approximated at rate $c\psi$ by the sequence (v_k) if $c < 1$.

The difficult point in the proof is to control also the quality of the approximations to x by rational points v that do not appear among the points v_k. Bugeaud's argument for that is based on continued fractions, and uses an elementary separation property for rational points on the real line: If v_1 and v_2 are two rational numbers with denominator at most q, then $d(v_1, v_2) \geq q^{-2}$. This property is of course also true for rational points in $\mathbb{R}^n$, $n \geq 2$, but