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ON THE FACES OF THE TENSOR CONE OF SYMMETRIZABLE KAC-MOODY LIE ALGEBRAS

BY Shrawan KUMAR AND Nicolas RESSAYRE

ABSTRACT. – In this paper, we are interested in the decomposition of the tensor product of two representations of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, or more precisely in the tensor cone of $\mathfrak{g}$. As usual, we parametrize the integrable, highest weight (irreducible) representations of $\mathfrak{g}$ by their highest weights. Then, the triples of such representations such that the last one is contained in the tensor product of the first two is a semigroup. This semigroup generates a rational convex cone $\Gamma(\mathfrak{g})$ called tensor cone. If $\mathfrak{g}$ is finite-dimensional, $\Gamma(\mathfrak{g})$ is a polyhedral convex cone. In 2006, Belkale and the first author described this cone by an explicit finite list of inequalities. In 2010, this list of inequalities was proved to be irredundant by the second author: each such inequality corresponds to a codimension one face. In general, $\Gamma(\mathfrak{g})$ is neither polyhedral nor closed. Brown and the first author obtained a list of inequalities that describe $\Gamma(\mathfrak{g})$ conjecturally. Here, we prove that each of these inequalities corresponds to a codimension one face of $\Gamma(\mathfrak{g})$.

RÉSUMÉ. – Dans cet article, nous nous intéressons à la décomposition du produit tensoriel de deux représentations d'une algèbre de Kac-Moody symétrisable $\mathfrak{g}$, et plus précisément au cône tensoriel de $\mathfrak{g}$. Comme d'habitude, nous paramétrons les représentations irréductibles intégrables et de plus haut poids par ledit plus haut poids. Alors, les triplets de telles représentations telles que la troisième s'injecte dans le produit tensoriel des deux premières est un semi-groupe. Ces triplets engendrent un cône convexe rationnel $\Gamma(\mathfrak{g})$ que nous appelons le *cône tensoriel*. Lorsque $\mathfrak{g}$ est de dimension finie, $\Gamma(\mathfrak{g})$ est un cône convexe polyédral. En 2006, Belkale et le premier auteur ont décrit ce cône par une liste finie explicite d'inégalités linéaires. En 2010, le second auteur a montré que cette liste d'inégalités n'est pas redondante : chaque inégalité correspond à une face de codimension un. En général, $\Gamma(\mathfrak{g})$ n'est ni fermé, ni polyédral. Brown et le premier auteur ont obtenu une liste d'inégalités qui décrit conjecturalement le cône $\Gamma(\mathfrak{g})$. Nous montrons ici que chacune de ces inégalités correspond à une face de codimension un de $\Gamma(\mathfrak{g})$.

1. Introduction

Let A be a symmetrizable irreducible GCM (generalized Cartan matrix) of size $l+1$. Let $\mathfrak{h} \supset \{\alpha_0^\vee, \dots, \alpha_l^\vee\}$ and $\mathfrak{h}^* \supset \{\alpha_0, \dots, \alpha_l\} =: \Delta$ be a realization of A over the complex

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numbers $\mathbb{C}$. We fix an integral form $\mathfrak{h}_{\mathbb{Z}} \subset \mathfrak{h}$ containing each $\alpha_i^\vee$, such that $\mathfrak{h}_{\mathbb{Z}}^* := \operatorname{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$ contains Δ and such that $\mathfrak{h}_{\mathbb{Z}}/\bigoplus_i \mathbb{Z}\alpha_i^\vee$ is torsion-free.

Set $\mathfrak{h}_{\mathbb{Q}}^* = \mathfrak{h}_{\mathbb{Z}}^* \otimes \mathbb{Q} \subset \mathfrak{h}^*$, $P_{+,\mathbb{Q}} := \{\lambda \in \mathfrak{h}_{\mathbb{Q}}^* : \langle \alpha_i^\vee, \lambda\rangle \geq 0 \quad \forall i\}$, and $P_+ := \mathfrak{h}_{\mathbb{Z}}^* \cap P_{+,\mathbb{Q}}$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac-Moody (KM) Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. For $\lambda \in P_+$, $L(\lambda)$ denotes the (irreducible) integrable, highest weight representation of $\mathfrak{g}$ with highest weight λ. Define the rational *tensor cone* as

$$\Gamma(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_{+,\mathbb{Q}}^3 : \exists N \geq 1 \text{ such that } L(N\mu) \subset L(N\lambda_1) \otimes L(N\lambda_2)\}.$$

The aim of this paper is to describe facets (codimension one faces) of this cone. Before describing our result, we recall from [4] a conjectural description of $\Gamma(\mathfrak{g})$, due to Brown and the first author. We need some more notation.

Fix $\{x_0, \dots, x_l\} \in \mathfrak{h}$ to be dual of the simple roots: $\langle \alpha_j, x_i\rangle = \delta_i^j$. Let $Q = \bigoplus_{i=0}^l \mathbb{Z}\alpha_i$ denote the root lattice. Let $X = G/B$ be the standard full KM-flag variety associated to $\mathfrak{g}$, where G is the 'minimal' Kac-Moody group with Lie algebra $\mathfrak{g}$ and B is the standard Borel subgroup of G. For w in the Weyl group W of G, let $X_w = \overline{BwB/B} \subset X$ be the corresponding Schubert variety. Let $\{\varepsilon^w\}_{w\in W} \subset \mathrm{H}^*(X, \mathbb{Z})$ be the (Schubert) basis dual (with respect to the standard pairing) to the basis of the singular homology of X given by the fundamental classes of X_w.

Let $P \supset B$ be a (standard) parabolic subgroup and let $X_P := G/P$ be the corresponding partial flag variety. Let W_P be the Weyl group of P (which is, by definition, the Weyl group of the Levi L of P) and let W^P be the set of minimal length representatives of cosets in W/W_P. The projection map $X \to X_P$ induces an injective homomorphism $\mathrm{H}^*(X_P, \mathbb{Z}) \to \mathrm{H}^*(X, \mathbb{Z})$ and $\mathrm{H}^*(X_P, \mathbb{Z})$ has the Schubert basis $\{\varepsilon_P^w\}_{w\in W^P}$ such that ε_P^w goes to ε^w for any $w \in W^P$. As defined by Belkale and the first author [1, §6] in the finite-dimensional case and extended by the first author in [13] for any symmetrizable Kac-Moody case (see [4, §,7] for more details), there is a new degenerated product $\odot_0$ in $\mathrm{H}^*(X_P, \mathbb{Z})$, which is commutative and associative. Now, we are ready to state Brown-Kumar's conjecture [4].

Conjecture 1.1. – *Let $\mathfrak{g}$ be any indecomposable symmetrizable Kac-Moody Lie algebra and let $(\lambda_1, \lambda_2, \mu) \in P_+^3$. Assume further that none of λ_j and μ are W-invariant and $\mu - \sum_{j=1}^2 \lambda_j \in Q$. Then, the following are equivalent:*

(a) $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$.

(b) *For every standard maximal parabolic subgroup P in G and every choice of triples $(w_1, w_2, v) \in (W^P)^3$ such that ε_P^v occurs with coefficient 1 in the degenerated product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in \big(\mathrm{H}^*(X_P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$(I^P_{w_1,w_2,v}) \qquad \lambda_1(w_1 x_P) + \lambda_2(w_2 x_P) - \mu(v x_P) \geq 0,$$

where α_{i_P} is the (unique) simple root not in the Levi of P and $x_P := x_{i_P}$.

Note that if λ_1 is W-invariant, $L(\lambda_1)$ is one-dimensional and hence $L(\lambda_1) \otimes L(\lambda_2)$ is irreducible.

In the case where $\mathfrak{g}$ is a semisimple Lie algebra, Conjecture 1.1 was proved by Belkale and the first author in [1]. The following result is due to the second author.

THEOREM 1.2 ([18]). – *In the case where* $\mathfrak{g}$ *is affine untwisted, Conjecture 1.1 holds.*

The conjecture in the general symmetrizable case is still open. But it is conceivable that the inductive proof in the case of affine $\mathfrak{g}$ obtained by the second author might be amenable to handle the general symmetrizable case.

Let us come back to the case where $\mathfrak{g}$ is semisimple. Then, $\Gamma(\mathfrak{g})$ is a closed convex polyhedral cone, and Conjecture 1.1, now known as Belkale-Kumar's theorem, describes $\Gamma(\mathfrak{g})$ in $(\mathfrak{h}_{\mathbb{Q}}^*)^3$ by (finitely many) explicit inequalities. Recall that a rational cone $\mathcal{C}$ is called *convex* if for $x, y \in \mathcal{C}$ and $0 < \alpha < 1, \alpha \in \mathbb{Q}, \alpha x + (1-\alpha)y \in \mathcal{C}$. In the case of $\mathfrak{g} = \mathfrak{sl}_n$, a larger set of inequalities describing $\Gamma(\mathfrak{g})$ was conjectured by Horn [7] and proved by Klyachko [9] (combining the saturation result of Knutson-Tao [10]). A larger set of inequalities describing $\Gamma(\mathfrak{g})$ for any semisimple $\mathfrak{g}$ was known earlier (see [3]). The irredundancy of the above set of inequalities $I^P_{w_1,w_2,v}$ was proved by Knutson-Tao-Woodward in type A [11] and by the second author in general [17]. See [14, §1] for more details on the history. The irredundancy assertion is the statement that each inequality $I^P_{w_1,w_2,v}$ in Conjecture 1.1 corresponds to a face of $\Gamma(\mathfrak{g})$ of codimension one. The aim of this paper is to extend this result to any symmetrizable Kac-Moody Lie algebra. We, in fact, prove the following stronger result for any, not necessarily maximal, standard parabolic subgroup P.

THEOREM 1.3. – *Let* $\mathfrak{g}$ *be any indecomposable symmetrizable Kac-Moody Lie algebra. Let* P *be a standard parabolic subgroup in* G *and let* $(w_1, w_2, v) \in (W^P)^3$ *be a triple such that* ε_P^v *occurs with coefficient* 1 *in the degenerated product*

$$\varepsilon_P^{w_1} \odot_0 \varepsilon_P^{w_2} \in \big(\mathrm{H}^*(X_P, \mathbb{Z}), \odot_0\big).$$

Then, the set of $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$ *such that for all* $\alpha_j \notin \Delta(P)$,

$$(I^j_{w_1,w_2,v}) \qquad \lambda_1(w_1 x_j) + \lambda_2(w_2 x_j) - \mu(v x_j) = 0$$

has codimension $\sharp(\Delta \backslash \Delta(P))$ *in* $\Gamma(\mathfrak{g})$, *where* $\Delta(P) \subset \Delta$ *is the set of simple roots of the Levi subgroup* L *of* P.

Let $\mathcal{C}$ denote the cone determined by the inequalities in Conjecture 1.1. For P maximal, Theorem 1.3 implies that if one removes any of the inequalities $I^P_{w_1,w_2,v}$, the cone thus obtained is strictly larger than $\mathcal{C}$.

Theorem 1.3 implies that $\mathcal{C}$ is locally polyhedral. This property of $\mathcal{C}$ plays an important role in the inductive proof of Theorem 1 from [18]. Note that in [18], the local polyhedrality is proved in a totally different way. As a consequence, one can hopefully think about Theorem 1.3 as a first step towards a proof of Conjecture 1.1.

Combining Theorems 1.2 and 1.3, we get the following.

COROLLARY 1.4. – *For any untwisted affine Kac-Moody Lie algebra* $\mathfrak{g}$, *the inequalities* $I^P_{w_1,w_2,v}$ *in Conjecture 1.1 give an irredundant and complete set of inequalities determining the cone* $\Gamma(\mathfrak{g})$.