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QUANTITATIVE CONDITIONS FOR RIGHT-HANDEDNESS OF FLOWS

BY ANNA FLORIO AND UMBERTO HRYNIEWICZ

ABSTRACT. – We give a numerical condition for right-handedness of a dynamically convex Reeb flow on the 3-sphere. Our condition is stated in terms of an asymptotic ratio between the amount of rotation of the linearised flow and the linking number of trajectories with a periodic orbit that spans a disk-like global surface of section. As an application, we find an explicit constant $\delta_* < 0.7225$ such that if a Riemannian metric on the 2-sphere is δ-pinched with $\delta > \delta_*$, then its geodesic flow lifts to a right-handed flow on the 3-sphere. In particular, all finite non-empty collections of periodic orbits of such a geodesic flow bind open books whose pages are global surfaces of section.

RÉSUMÉ. – Nous donnons une condition numérique garantissant qu'un flot de Reeb dynamiquement convexe sur la sphère S^3 soit lévogyre. Notre condition fait intervenir un certain rapport asymptotique entre le taux de rotation du flot linéarisé et le nombre d'enlacement entre les trajectoires et une orbite périodique qui borde une surface de section globale de type disque. En application, nous trouvons une constante explicite $\delta_* < 0{,}7225$ telle que si une métrique riemannienne sur la sphère S^2 est δ-pincée avec $\delta > \delta_*$, alors son flot géodésique se relève en un flot lévogyre sur la sphère S^3. En particulier, toute collection non vide finie d'orbites périodiques d'un tel flot géodésique borde un livre ouvert dont les pages sont des surfaces de section globale.

1. Introduction

Right-handed vector fields, introduced by Ghys in [10], form a special class of non-singular vector fields on homology 3-spheres. Their flows will also be called right-handed, for simplicity. Roughly speaking, all pairs of trajectories of such a flow link positively. Ghys formalized this definition in terms of positivity of the quadratic linking form, which assigns some kind of linking number to any pair of invariant Borel probability measures; see

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also [2]. Right-handedness has interesting dynamical implications as the following statement demonstrates.

THEOREM 1.1 (Ghys [10]). – *Every non-empty finite collection of periodic orbits of a right-handed flow binds an open book whose pages are global surfaces of section.*

In particular, the above statement imposes strong restrictions on the periodic orbits that can arise in right-handed flows: knots or links that are not fibred cannot be realized. Both the definition of right-handedness (Definition 2.6) and the proof of Theorem 1.1 ([9, Appendix B]) can actually be given independently of the quadratic linking form.

Right-handedness is difficult to check. It can be checked for certain special flows, like the Hopf flow or other integrable flows. Since in [10] it is stated that right-handedness is a C^1-open condition, one knows that there exist more interesting right-handed flows near these simple examples, but this abstract reasoning does not give them explicitly. It is precisely this general lack of explicit examples that motivates our work. One exception is provided by the work of Dehornoy in [5]. Sometimes it is more natural to talk about left-handedness, which is equivalent to right-handedness when the ambient orientation is reversed. Dehornoy showed that the geodesic flow on the unit tangent bundle of a hyperbolic n-conic 2-sphere is left-handed if, and only if, $n = 3$. Then, using a version of Gromov's geodesic rigidity [11], the case of arbitrary negative sectional curvature is reduced to the hyperbolic case.

To state our main application, consider the polynomial $P(x) = 4x^3 - 2x^2 - 1$, which has a unique real root x_*. It satisfies $0.84 < x_* < 0.85$. Set $\delta_* := x_*^2$. Given $\delta \in (0, 1]$, a Riemannian metric on S^2 is said *δ-pinched* if $\delta \leq K_{\min}/K_{\max}$, where $K_{\min}, K_{\max}$ denote the minimum and the maximum of the Gaussian curvature.

THEOREM 1.2. – *If $\delta > \delta_*$, in particular if $\delta \geq 0.7225$, then the geodesic flow of a δ-pinched Riemannian metric on S^2 lifts to a right-handed flow on S^3.*

As mentioned before, the main motivation for such a statement is that it is not a perturbative result, hence it can be used to check right-handedness in an explicit set of flows that are far from integrable.

Other implications of right-handedness besides Theorem 1.1 have been obtained. For instance, Dehornoy and Rechtman showed in [6] that if an invariant measure of a right-handed flow can be approximated by long periodic orbits, then their Seifert genera grow proportionally to the square of the period, the proportionality constant being the helicity of the given invariant measure.

More generally, we look for numerical conditions for right-handedness within the class of Reeb flows of dynamically convex contact forms on S^3. This class was introduced by Hofer, Wysocki and Zehnder in [14]. They showed that this class is rich enough to provide applications: by [14, Theorem 3.4] the Hamiltonian flow on a strictly convex energy level in a 4-dimensional symplectic vector space is the Reeb flow of a dynamically convex contact form. Harris and Paternain showed in [13] that the geodesic flow of a Finsler metric on the two-sphere with reversibility r is dynamically convex if the flag curvatures are pinched by more than $(r/(r+1))^2$. Theorem 1.2 will be deduced from Theorem 1.13 below, which provides an abstract condition for right-handedness of a dynamically convex Reeb flow on S^3 in terms of a dynamical pinching condition. Theorem 1.13 also implies Theorem 1.14 which gives right-handedness on strictly convex energy levels in terms of a relation between the curvatures and the return times of a disk-like global surface of section.

It was explained to us by Dehornoy in private communication that there are Riemannian metrics on S^2 with strict positive curvature whose geodesic flows do not lift to right-handed flows on S^3. Consider the infimum δ_0 of all $\delta \in (0,1]$ with the following property: if a Riemannian metric on S^2 has sectional curvature pinched by at least δ then its geodesic flow lifts to a right-handed flow on S^3. Dehornoy's examples show that $\delta_0 > 0$. Together with Theorem 1.2 we get $0 < \delta_0 \leq \delta_* < 0.7225$. We are led to ask:

Question. What is the value of δ_0?

1.1. Global surfaces of section

Let X be a smooth vector field on a closed and oriented 3-manifold M. Its flow is denoted by ϕ^t.

Definition 1.3. – A *global surface of section* for ϕ^t, or for X, is a smooth, embedded and compact surface $\Sigma \hookrightarrow M$, such that $\partial\Sigma$ consists of periodic orbits or is empty, X is transverse to $\Sigma \setminus \partial\Sigma$, and for every $x \in M$ one finds $t_- < 0 < t_+$ such that $\phi^{t_\pm}(x) \in \Sigma$.

Remark 1.4. – The orientation of M and the co-orientation of $\Sigma \setminus \partial\Sigma$ induced by X together orient Σ. In this paper we always assume that global surfaces of section are oriented in this way.

Remark 1.5. – From a dynamical perspective a global surface of section is a valuable tool since one can deduce dynamical properties of the flow from those of the associated *first return map*. To define it we need first to consider the *return time function* $\tau : \Sigma \setminus \partial\Sigma \to (0, +\infty)$, $\tau(x) = \min\{t > 0 \mid \phi^t(x) \in \Sigma\}$. One can use τ to define the return map $\psi : \Sigma \setminus \partial\Sigma \to \Sigma \setminus \partial\Sigma$ by $\psi(x) = \phi^{\tau(x)}(x)$. It follows from Definition 1.3 that ψ is a smooth diffeomorphism.

Consider the normal bundle to the flow $\xi = TM/\mathbb{R}X \to M$, and denote by $\mathbb{P}_+\xi$ the circle bundle $(\xi \setminus 0)/\mathbb{R}_+ \to M$. The latter is isomorphic to the unit normal bundle in ξ once a choice of metric is fixed. The equivalence class in $\mathbb{P}_+\xi$ of a non-zero vector $v \in \xi$ will be denoted by $\mathbb{R}_+v$. The linearised flow $D\phi^t$ induces flows on ξ and on $\mathbb{P}_+\xi$, both denoted by $D\phi^t$ with no fear of ambiguity. These flows cover ϕ^t. Both ξ and $\mathbb{P}_+\xi$ get oriented as bundles by the flow and the ambient orientation. Now let $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ be a periodic orbit of ϕ^t, where $T > 0$ is the primitive period. The total space $\mathbb{T}_\gamma$ of the trivial circle bundle $\gamma(T\cdot)^*\mathbb{P}_+\xi \to \mathbb{R}/\mathbb{Z}$, which we see as a submanifold of $\mathbb{P}_+\xi$, is a $D\phi^t$-invariant torus. The dynamics of $D\phi^t$ on $\mathbb{T}_\gamma$ will be referred to as *linearised polar dynamics* along γ. For each boundary orbit γ of a global surface of section Σ, consider

$$\nu^\Sigma\gamma = \{\mathbb{R}_+v \mid v \in T\Sigma|_\gamma \text{ is outward pointing}\}.$$

It follows that $\nu^\Sigma\gamma/\mathbb{R}X$ defines the graph of a section of $\gamma(T\cdot)^*\mathbb{P}_+\xi$ and, as such, determines a smooth submanifold of $\mathbb{T}_\gamma$.