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ON THE LONG-TIME BEHAVIOR OF SCALE-INVARIANT SOLUTIONS TO THE 2D EULER EQUATION AND APPLICATIONS

BY AYMAN R. SAID, TAREK M. ELGINDI AND RYAN W. MURRAY

ABSTRACT. – We study the long-time behavior of scale-invariant solutions of the 2d Euler equation satisfying a discrete symmetry. We show that all scale-invariant solutions with bounded variation on $\mathbb{S}^1$ relax to states that are piecewise constant with finitely many jumps. All continuous scale-invariant solutions become singular and homogenize in infinite time. On $\mathbb{R}^2$, this corresponds to generic infinite-time spiral and cusp formation. The main tool in our analysis is the discovery of a monotone quantity that measures the number of particles that are moving away from the origin. This monotonicity also applies locally to solutions of the 2d Euler equation that are m-fold symmetric ($m \geq 4$) and have radial limits at the point of symmetry.

Our results are also applicable to the Euler equation on a large class of surfaces of revolution (such as $\mathbb{S}^2$ and $\mathbb{T}^2$). Our analysis then gives generic spiraling of trajectories and infinite-time loss of regularity for globally smooth solutions on any such smooth surface, under a discrete symmetry.

RÉSUMÉ. – Nous étudions le comportement en temps long des solutions des équations d'Euler en dimension deux qui sont invariantes par changement d'échelle admettant une symétrie par rotation d'un angle fixe $\frac{2\pi}{m}$, $m \in \mathbb{N}$. Nous montrons que toute solution à variation bornée sur $\mathbb{S}^1$ converge en temps infini vers une solution constante par morceaux admettant un nombre fini de sauts. Toute solution continue devient singulière et s'homogénéise en temps infini. Sur $\mathbb{R}^2$ ceci correspond à une formation générique de spirales et de points de rebroussement. L'outil principal de notre analyse est la découverte d'une fonctionnelle monotone qui mesure le nombre de particules sortant de l'origine. Cette monotonicité s'applique plus généralement, localement, à toutes les solutions des équations d'Euler en deux dimensions de l'espace qui ont une symétrie par rotation d'un angle fixe $\frac{2\pi}{m}$, $m \geq 4$ et qui admettent des limites radiales au point de symétrie.

Notre analyse s'applique également aux équations d'Euler sur une classe large de surfaces de rotation (par exemple $\mathbb{S}^2$ et $\mathbb{T}^2$). Notre analyse montre alors que génériquement les trajectoires des particules décrivent des spirales et qu'il y a génériquement une perte de régularité en temps infini pour des solutions globalement régulières sur une telle surface lisse sous la symétrie discrète.

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1. Introduction

We are concerned with 2d inviscid flows; namely, solutions to the incompressible Euler equation:

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1.1}$$

$$u = \nabla^{\perp} \Delta^{-1} \omega. \tag{1.2}$$

Here, the scalar vorticity $\omega : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is transported by the velocity field $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, which is uniquely determined at each time $t \in \mathbb{R}$ from ω using the Newtonian potential:

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2} \omega(y,t) dy. \tag{1.3}$$

We adopt the standard notation $v^{\perp} = (-v_2, v_1)$ for $v = (v_1, v_2) \in \mathbb{R}^2$. It is well known that smooth enough solutions to the 2d Euler Equation (1.1)-(1.2) retain their smoothness for all finite times. Much less is known in the infinite-time limit. Since the Euler equation is fundamentally a (non-linear and non-local) transport equation, there is a strong possibility that despite the plethora of possible initial states, most solutions "relax" in infinite time to simpler states. This has been established in perturbative regimes in the ground breaking work of Bedrossian and Masmoudi [1] and later extensions by Ionescu and Jia [11, 12] and Masmoudi and Zhao [17]. As for the generic long-time behavior of solutions, there are two natural conjectures regarding the long time behavior of solutions to the 2d Euler equation; see [26] and [22] respectively and also the review articles [6, 14]. We state these conjectures on $\mathbb{T}^2$ for convenience:

CONJECTURE 1.1. – *As $t \to \pm\infty$, generic solutions experience loss of compactness in L^2.*

CONJECTURE 1.2. – *The (weak) limit set of generic solutions consists only of solutions lying on compact orbits.*

These two conjectures together state that most solutions should, on the one hand, "relax" in infinite time in that they should lose L^2 mass, due to mixing [4]. Note that this is the only way that compactness can be lost on compact domains since the L^2 norm of the vorticity is conserved for all finite times. On the other hand, these limiting states are conjectured to have compact orbits; i.e., they must be very special, such as steady states, time-periodic solutions, etc. While there appear to be no results in the literature proving either of these phenomena in large data settings, there are a few results on generic small scale creation. Of note is the result of Koch [16] in which strong growth of Hölder and Sobolev norms of the vorticity is established near any background solution (stationary or time-dependent) for which the gradient of the flow map is unbounded in time. Yudovich also established (boundary induced) growth results under some mild assumption on the data near the boundary of the domain [27] (see also [18] for an extension of [27]). There are also numerous important results on growth of solutions in the neighborhood of stable steady states [19, 5, 15, 28, 6].

The purpose of this work is to exhibit one setting where generic relaxation and growth can be established rigorously and in full generality and, in particular, away from equilibrium. We do this in the setting of scale-invariant solutions, introduced in [8], which are only assumed to have bounded vorticity. We find that, buried in the dynamics of these Euler solutions, there is a powerful relaxation mechanism that induces both an arrow of time and a major contraction of phase space. Our main theorems can be stated informally as

THEOREM 1. – *Consider the 2d Euler equation on* $\mathbb{R}^2$.

- *The set of* C^1 *and m-fold symmetric initial data whose corresponding solution is unbounded in* C^1 *(as* $t \to \infty$*) is dense, within the family of m-fold symmetric solutions. Moreover, particle trajectories generically form spirals in infinite time.*
- *Scale-invariant solutions relax in infinite time, by virtue of a monotone quantity, to states with finitely many jump discontinuities.*

The first statement also holds on $\mathbb{S}^2$ and requires $m \geq 3$, while the second requires $m \geq 4$. Note that the first statement applies to *all* smooth solutions obeying the discrete symmetry, not just scale-invariant solutions. To the best of our knowledge, these are the only available large data results on generic solutions to the 2d Euler equation (even within symmetry).

To properly frame the discussion, we begin by recalling the definition and properties of scale-invariant solutions.

1.1. Scale-invariant solutions

Solutions to the 2d Euler equation enjoy a two-parameter family of scaling symmetries. Indeed, once ω solves (1.1)-(1.2), we have that $\omega_{\lambda,\mu}$ defined by

$$\omega_{\lambda,\mu}(t,x) = \frac{1}{\mu}\omega(\mu t, \lambda x)$$

also solves the Euler equation whenever $\mu, \lambda \in (0,\infty)$. It is natural to consider solutions that are scale-invariant:

DEFINITION 1.3. – *The map* $\omega : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ *is said to be* scale-invariant *if* $\omega(\cdot, \lambda x) = \omega(\cdot, x)$ *for all* $\lambda \in (0,\infty)$ *and* $x \in \mathbb{R}^2$.

An unfortunate fact about non-trivial scale-invariant solutions is that they cannot decay at spatial infinity. This makes it challenging to make sense of (1.3). A key observation from [8] was that we could give a rigorous meaning to (1.3) when the vorticity ω satisfies a discrete symmetry. Indeed, in the most extreme case, when ω is radially symmetric, the Formula (1.3) becomes completely local and *does not depend in any way* on the behavior of ω as $|x| \to \infty$. This is also true in an asymptotic sense when the vorticity satisfies a discrete rotational symmetry, and it is a fortunate fact that discrete rotational symmetries are propagated by the Euler equation. Indeed, if $\omega(x,t)$ solves the 2d Euler equation then whenever $\mathcal{O} \in \mathrm{SO}(2)$, $\omega_{\mathcal{O}}$ defined by

$$\omega_{\mathcal{O}}(t,x) = \omega(t, \mathcal{O}x),$$

is also a solution. We now give the precise definition of discrete symmetry:

DEFINITION 1.4. – *For* $m \in \mathbb{N}$ *a function* $\omega : \mathbb{R}^2 \to \mathbb{R}$ *is said to be m-fold symmetric if* $\omega(\mathcal{O}_m x) = \omega(x)$, *for all* $x \in \mathbb{R}^2$, *where* $\mathcal{O}_m \in \mathrm{SO}(2)$ *is the matrix corresponding to a counterclockwise rotation by angle* $\frac{2\pi}{m}$ *about the origin.*

If X is a space of functions on $\mathbb{R}^2$, we will denote by X_m the space of functions in X that are m-fold symmetric. We will now state the main theorem of [8].