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FLEXIBILITY OF THE ADJOINT ACTION OF THE GROUP OF HAMILTONIAN DIFFEOMORPHISMS

BY Lev BUHOVSKY AND Maksim STOKIĆ

ABSTRACT. – On a closed and connected symplectic manifold, the group of Hamiltonian diffeomorphisms has the structure of an infinite-dimensional Fréchet Lie group, where the Lie algebra is naturally identified with the space of smooth and zero-mean normalized functions, and the adjoint action is given by pullbacks. We show that this action is flexible: for every non-zero smooth and zero-mean normalized function u, any other smooth and zero-mean function f can be written as a finite sum of elements in the orbit of u under the adjoint action. Additionally, the number of elements in this sum is dominated by the uniform norm of f. This result can be interpreted as a (bounded) infinitesimal version of Banyaga's theorem on the simplicity of the group of Hamiltonian diffeomorphisms.

RÉSUMÉ. – Sur une variété symplectique compacte et connexe, le groupe des difféomorphismes hamiltoniens possède la structure d'un groupe de Lie de Fréchet de dimension infinie, dont l'algèbre de Lie s'identifie naturellement à l'espace des fonctions lisses normalisées de moyenne nulle, et l'action adjointe est par tirés en arrière. Nous démontrons que cette action est flexible : pour chaque fonction lisse non nulle, normalisée et de moyenne nulle u, toute autre fonction lisse, et de moyenne nulle f peut être écrite comme une somme finie d'éléments de l'orbite de u sous l'action adjointe. De plus, le nombre d'éléments dans cette somme est dominé par la norme uniforme de f. Ce résultat peut être interprété comme une version infinitésimale (bornée) du théorème de Banyaga sur la simplicité du groupe des difféomorphismes hamiltoniens.

1. Introduction and main results

Consider a closed and connected symplectic manifold (M, ω) of dimension $2n$. Let $C_0^\infty(M)$ be the space of smooth functions that are zero-mean normalized with respect to the volume form ω^n. When equipped with the C^∞-topology, the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ is an infinite-dimensional Fréchet Lie group, whose Lie algebra $\mathcal{A}$ can be identified with the space $C_0^\infty(M)$. The adjoint action of $\mathrm{Ham}(M, \omega)$ on $\mathcal{A}$ is given by $\mathrm{Ad}_\phi f = f \circ \phi^{-1}$, for every $f \in \mathcal{A} = C_0^\infty(M)$ and $\phi \in \mathrm{Ham}(M, \omega)$. Our main result shows flexibility of the adjoint action in the following sense:

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THEOREM 1. – *Let (M, ω) be a closed and connected symplectic manifold, and let $u \in C_0^\infty(M)$ be a non-zero function. There exists $N \in \mathbb{N}$ that only depends on u, such that for any $f \in C_0^\infty(M)$ with $\|f\|_\infty \leq 1$, one can write*

$$f = \sum_{i=1}^{N} \Phi_i^* u$$

for some Hamiltonian diffeomorphisms $\Phi_i \in \mathrm{Ham}(M, \omega)$.

For a non-zero function $u \in C_0^\infty(M)$, denote by $N(u) \in \mathbb{N}$ the *minimal* N (the number of summands) as in the theorem. Then $N(u)$ is invariant under the action of symplectic diffeomorphisms, and it would be interesting to understand its properties better.

One can view Theorem 1 as an infinitesimal analogue of the simplicity of the group $\mathrm{Ham}(M, \omega)$ proved by Banyaga [2]. Indeed, simplicity of $\mathrm{Ham}(M, \omega)$ is equivalent to saying that for any non-trivial $\Phi \in \mathrm{Ham}(M, \omega)$, any other $\Psi \in \mathrm{Ham}(M, \omega)$ can be written as

$$\Psi = \prod_{i=1}^{m} \Theta_i^{-1} \Phi^{\pm 1} \Theta_i, \tag{1}$$

where $\Theta_i \in \mathrm{Ham}(M, \omega)$ for each i. Now, if we fix an autonomous Hamiltonian function H and assume that Φ^ε is the time-ε map of the Hamiltonian flow of H (when ε is small), then up to $o(\varepsilon)$ the Hamiltonian diffeomorphism $\Psi^\varepsilon = \prod_{i=1}^{m} \Theta_i^{-1} (\Phi^\varepsilon)^{\pm 1} \Theta_i$ equals to the time-ε map of the Hamiltonian flow generated by $F = \sum_{i=1}^{m} \pm \Theta_i^* H$. Note that in Theorem 1 subtraction is not needed, and only addition is used for representing the function f via pullbacks of u by Hamiltonian diffeomorphisms. Moreover, Theorem 1 guarantees existence of such a representation where the number of terms does not depend on the function f (provided that $\|f\|_\infty \leq 1$).

REMARK 1.1. – Comparing our result with Banyaga's simplicity theorem, one may ask if the number of terms m in the representation (1) is bounded from above by some number m_0, provided that $\|\Psi\|_{\mathrm{Hofer}} \leq 1$. The assumption on the Hofer's norm of Ψ is necessary, since the triangle inequality implies $\|\Psi\|_{\mathrm{Hofer}} \leq m \cdot \|\Phi\|_{\mathrm{Hofer}}$. However, this assumption is not enough to bound m. A function $r : \mathrm{Ham}(M, \omega) \to \mathbb{R}$ is called a *homogeneous quasimorphism* if there exists a constant D such that $|r(\phi \circ \psi) - r(\phi) - r(\psi)| \leq D$ and $r(\phi^m) = m r(\phi)$ for all $\phi, \psi \in \mathrm{Ham}(M, \omega)$ and $m \in \mathbb{Z}$. If a homogeneous quasimorphism on $\mathrm{Ham}(M, \omega)$ exists, and m is bounded by m_0, we get

$$\|\Psi\|_{\mathrm{Hofer}} \leq 1 \implies |r(\Psi)| \leq C := (m_0 - 1)D + m_0 \cdot |r(\Phi)|.$$

Now let $\Psi \in \mathrm{Ham}(M, \omega)$ with $\|\Psi\|_{\mathrm{Hofer}} \leq 1/2$, and let $k = \lfloor 1/\|\Psi\|_{\mathrm{Hofer}} \rfloor$. Note that $\|\Psi^k\|_{\mathrm{Hofer}} \leq k \cdot \|\Psi\|_{\mathrm{Hofer}} \leq 1$, therefore we can apply the previous estimate for Ψ^k and get

$$k \cdot |r(\Psi)| = |r(\Psi^k)| \leq C.$$

From here we conclude

$$\|\Psi\|_{\mathrm{Hofer}} \leq 1/2 \implies |r(\Psi)| \leq 2C \|\Psi\|_{\mathrm{Hofer}}.$$

In particular, we get the Hofer continuity of r at the identity. However, there are examples of symplectic manifolds and homogeneous quasimorphisms that are not Hofer continuous at the identity (see [7] for examples).

REMARK 1.2. – Despite the simplicity of $\mathrm{Ham}(M,\omega)$ and Theorem 1, the Lie algebra $\mathcal{A}$ is not simple (this contrasts with the case of finite-dimensional Lie groups). Indeed, for any open subset $U \subset M$, the space of all $f \in C_0^\infty(M)$ that vanish on U forms an ideal of $\mathcal{A}$ which is nontrivial and proper provided that $U \neq \emptyset$ and that $M \setminus U$ has a non-empty interior.

In [5] Buhovsky and Ostrover showed the following result, which was also recently reproved by Lempert [9] via an elegant functional analytic approach:

THEOREM A (Buhovsky-Ostrover [5]; Lempert [9]). – Let (M,ω) be a closed symplectic manifold. Any $\mathrm{Ham}(M,\omega)$-invariant pseudo-norm $\|\cdot\|$ on $\mathcal{A}$ that is continuous in the C^∞-topology, is dominated from above by the L_∞-norm i.e., $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant C.

Theorem 1 readily implies that one can remove the condition of continuity in C^∞ topology:

THEOREM 2. – *Let (M,ω) be a closed symplectic manifold. Any $\mathrm{Ham}(M,\omega)$-invariant norm on the space $\mathcal{A} = C_0^\infty(M)$ is bounded from above by a constant multiple of the L_∞-norm.*

Proof. – Without loss of generality we can assume that M is connected. Fix a non-zero function $u \in C_0^\infty(M)$, and let $f \in C_0^\infty(M)$. Theorem 1 gives us a representation

$$f = \|f\|_\infty \cdot \textstyle\sum_{i=1}^N \Phi_i^* u,$$

where N depends only on u. Applying the triangle inequality and $\mathrm{Ham}(M,\omega)$-invariance of the norm $\|\cdot\|$ we get $\|f\| \leq N \cdot \|u\| \cdot \|f\|_\infty$. □

One ingredient in the proof of Theorem 1 is a property of solutions of the equation $h(t) = f(t+\alpha) - f(t)$, where $h : S^1 \to \mathbb{R}$ is a given smooth function of zero mean, $\alpha \in \mathbb{R}$ is an irrational number which is badly approximable by rationals, and $f : S^1 \to \mathbb{R}$ is an unknown function whose properties are of interest (see Lemma 2.6 in Section 2.1 below). We remark that this equation is a simplest example of a small denominators problem [1], and its study already appeared in Hilbert's book [6, Chapter 17, Section 5]. Equations of that type also play an important role in establishing simplicity of classical diffeomorphisms groups [3], however, interestingly enough, the way the equation is used in our proof of Theorem 1, seems to be different.

In [12], Polterovich proved the following remarkable averaging property:

THEOREM B (Polterovich [12]). – Let (M,ω) be a closed and connected symplectic manifold, and let $u \in C(M)$ be a continuous function which is zero-mean normalized with respect to the volume form ω^n. Then for every $\varepsilon > 0$ there exist $\Phi_1, \ldots, \Phi_N \in \mathrm{Ham}(M,\omega)$ such that

$$\tfrac{1}{N}|u \circ \Phi_1(x) + \cdots + u \circ \Phi_N(x)| < \varepsilon$$

for every $x \in M$.

In the case of an open connected symplectic manifold such a statement holds as well (for compactly supported u) and plays an important role in the proof of Theorem 1. But in fact, for a *closed* connected symplectic manifold (M,ω) and a *smooth* non-zero function $u \in C_0^\infty(M)$, Theorem 1 implies a sharp version of Theorem C, since in particular it shows that *the zero function can be represented as such a sum of pullbacks of u by Hamiltonian diffeomorphisms*. It is natural to ask whether this sharp version of Theorem C (or more